Lecture Notes on Bihamiltonian Structures and their Central Invariants

Si-Qi Liu

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0 Introduction

Let X be a toric Fano variety. It is well known that the quantum cohomology $QH^*(X)$ is a semisimple Frobenius manifold, and the generating function of all its Gromov-Witten invariants, which is usually called the total descendant potential of X, is given by Givental's quantization formula (see [25] for more details):

$$Z_X = \tau_I(u)\hat{S}_u^{-1}(z)\hat{\Psi}_u\hat{R}_u(z)e^{(U/z)\hat{}}\left(\prod_{i=1}^n Z_{pt}^{(i)}\right).$$

Givental also proved the Virasoro conjecture in this case, that is, Z_X satisfies the Virasoro constraints

$$L_m Z_X = 0, \quad m \geq -1,$$

where $\{L_m\}_{m\in\mathbb{Z}}$ is a set of linear differential operators satisfying the Virasoro commuting relations [17, 22].

The preprint version of Givental's [25] was released in Aug 2001. In the same month, Dubrovin and Zhang put another preprint [15] on arXiv, which showed that the total descendant potential of a semisimple Frobenius manifold is uniquely determined by its genus zero part and the Virasoro constraints. Let $F = \log Z_X$ be the free energy of X, then expand F with respect to the string coupling constant $\hbar$

$$F = \sum_{g\geq 0} \hbar^{g-1} F_g.$$

Dubrovin and Zhang derived a series of differential equations for F_g from the Virasoro constraints, whose generating function is called the loop equation for X, and showed that one can obtained F_g recursively from these equations. In particular, they gave an explicit formula of F_2 for an arbitrary semisimple Frobenius manifold, which is not easy to obtain from Givental's quantization formula.

According to Dubrovin-Zhang's uniqueness theorem, their approach is equivalent to Givental's quantization formula. Givental's formula has drawn much attention, while Dubrovin-Zhang's approach is less well known. One possible reason is that Dubrovin-Zhang's preprint [15] is too long: it contains more than 180 pages, whose first 150 pages are about an axiomatic framework for integrable systems that may govern a Gromov-Witten theory. Their loop equation appears in the last 30 pages, and the main results are also proved in this last part. It seems that to understand their main results one must read the first 150 pages, which is indeed a tough work for people not working on integrable systems. But in my personal opinion, the last 30 pages of Dubrovin-Zhang's preprint is almost independent of the first 150 pages, so one can read it directly.

In an informal workshop on Landau-Ginzburg B-model held in University of Michigan, Mar 10–14, 2014, I gave a short introduction to Dubrovin-Zhang's loop equation, especially on the case with X = point. I planned to give more details for general cases in the present lecture notes. But Zhang told me that Dubrovin and he have been working on a similar introductory paper for months, and there is also a good introduction to this subject in Dubrovin's new paper [8], so I decide to talk about something else – something on the first 150 pages of Dubrovin and Zhang's preprint [15].

Saying one can skip the first 150 pages of [15] doesn't mean that this part is not important. Instead, this part is more general, so it includes not only the cases in which Givental's formula or Dubrovin-Zhang's loop equation work but also the cases make these two approaches fail. For example, Dubrovin-Zhang's axiom system consists of four axioms (see [15] for more details):

- BH=Bihamiltonian structure

- QT=Quasi-triviality
- TS=Tau structure
- VS=Linearizable Virasoro symmetries

If an integrable system satisfies all these axioms, the corresponding total descendant potential must be given by Givental's formula or Dubrovin-Zhang's loop equation. But, if it satisfies all but the last axiom, one can also define its total descendant potential, and this potential is not equivalent to Givental's one in general. Recently, Wu showed that the Drinfeld-Sokolov hierarchies of BCFG types are integrable systems of this kind [34]. Then Ruan, Zhang and I show that the generating functions of FJRW invariants of boundary singularities of BCFG type gives tau functions of these integrable systems [27]. In particular, we show that the BCFG Drinfeld-Sokolov hierarchies must be not equivalent to Dubrovin-Zhang's hierarchies, so the generating function of BCFG FJRW invariants must be not given by Givental's formula.

To show that two integrable systems are not equivalent is highly nontrivial. One need to find out the orbits of a class of integrable systems under the action of a certain transformation group. Such a classification problem is first precisely stated in Dubrovin-Zhang's [15] for the integrable systems satisfying the BH axiom. We introduced the concept of *central invariants*, which can be regarded as coordinates on the orbit space, and answered the uniqueness part of this classification problem [28, 10]. As a byproduct, we also show that the QT axiom is a corollary of the BH axiom, which is also conjectured and partially proved in [15]. The existence part of the above classification problem is also resolved recently. In [30], we founded a new framework for the computation of the cooresponding bihamiltonian cohomologies, and proved the existence theorem for the simplest case, that is the bihamiltonian structure of the Korteweg-de Vries hierarchy. We planned to consider the general cases in [12] by using a similar argument. This is not an easy generalization, because our computation method, even for the simplified one, is still very complicated. In a recent preprint [2] (c.f. [1]), Carlet, Posthuma, and Shadrin developed some new computing techniques based on our approach, several interesting spectral sequences, and some homotopy formulae, then proved the existence theorem for the general cases.

The central invariants of a bihamiltonian structure are a set of functions of one variable. For the integrable systems satisfying Dubrovin-Zhang's four axioms, all central invariants must be $1/24$. On the other hand, we computed the central invariants for the bihamiltonian structure for Drinfeld-Sokolov hierarchies [11]. For the BCFG cases, their central invariants are unequal constants, so they are not equivalent to Dubrovin-Zhang's integrable hierarchies.

In these lecture notes, I will give an introduction to our results with as much details as possible. In Section 1, I recall some basic facts of finite dimensional Poisson geometry. We introduce the Schouten-Nijenhuis bracket in an unusual way, which can be also used in the infinite dimensional case. Then we give the definition of Hamiltonian structures for partial differential equations in Section 2. In Section 3 and 4, we prove some results on the relation between classification problems of (bi)hamiltonian structures and their cohomologies. We also prove a Darboux theorem for certain Hamiltonian structures. Then we introduce the notion of central invariants of a semisimple bihamiltonian structure in

Section 5. In the last subsection, we give an introduction to the Drinfeld-Sokolov bihamiltonian structure and their central invariants.

1 Finite dimensional Poisson geometry

1.1 Basic definition

Let M be a smooth manifold of dimension n, and $\mathcal{A}_0 = C^\infty(M)$ be the algebra of smooth functions on M (we will explain why we use this notation in the next section). A Poisson bracket on M is, by definition, a bilinear map $\{\,,\}$: $\mathcal{A}_0 \times \mathcal{A}_0 \to \mathcal{A}_0$ satisfying the following conditions:

$$\text{Skew-symmetry: } \{f, g\} + \{g, f\} = 0, \tag{1.1}$$

$$\text{Jacobi identity: } \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0, \tag{1.2}$$

$$\text{Leibniz's rule: } \{f \cdot g, h\} = f \cdot \{g, h\} + \{f, h\} \cdot g, \tag{1.3}$$

where $f, g, h \in \mathcal{A}_0$, and $\cdot$ is the multiplication of $\mathcal{A}_0$. The manifold M is called a Poisson manifold if it is equipped with a Poisson bracket.

The condition (1.1) and (1.2) show that $(\mathcal{A}_0,\ \{\,,\})$ forms a Lie algebra, and the condition (1.3) implies (by using Hadamard's Lemma) that the Poisson bracket is locally given by [1]

$$\{f, g\} = P^{\alpha\beta}(u)\frac{\partial f}{\partial u^\alpha}\frac{\partial g}{\partial u^\beta}, \tag{1.4}$$

where $(u^1, \dots, u^n)$ is a set of local coordinates on M. The functions $P^{\alpha\beta}(u)$ are actually given by

$$\{u^\alpha, u^\beta\} = P^{\alpha\beta}(u),$$

and they are called the components of the Poisson bracket $\{\,,\}$ in the local coordinates system $(u^1, \dots, u^n)$.

The formula (1.4) shows that we can introduce a bivector, i.e. a skew-symmetric tensor of $(2, 0)$ type,

$$P = P^{\alpha\beta}(u)\frac{\partial}{\partial u^\alpha} \wedge \frac{\partial}{\partial u^\beta}, \tag{1.5}$$

and then write the Poisson bracket as the following form

$$\{f, g\} = \langle P, \mathrm{d}f \wedge \mathrm{d}g\rangle,$$

where $\langle\,,\rangle$ is the standard pairing between tensors of $(2, 0)$ and $(0, 2)$ types. The tensor P is called the Poisson tensor or Poisson structure of the Poisson manifold $(M,\ \{\,,\})$.

The condition (1.2) of the Poisson bracket $\{\,,\}$ is equivalent to the following condition on the components of P:

$$\frac{\partial P^{\alpha\beta}}{\partial u^\sigma}P^{\sigma\gamma} + \frac{\partial P^{\beta\gamma}}{\partial u^\sigma}P^{\sigma\alpha} + \frac{\partial P^{\gamma\alpha}}{\partial u^\sigma}P^{\sigma\beta} = 0. \tag{1.6}$$

[1] In this paper, summation over repeated *Greek* indexes is always assumed, and we don't sum over *Latin* indexes.

This condition also has a coordinate-free form, which requires the notion of Schouten-Nijenhuis bracket.

The Schouten-Nijenhuis bracket is a bilinear operation defined on the space $\Lambda^* = \Gamma(\wedge^* T(M))$ of polyvectors. There are several equivalent ways to define this operation. We give two of them, which can be easily generalized to the infinite-dimensional case.

1.2 Nijenhuis-Richardson bracket

Let $P \in \Lambda^p$ be a p-vector. We define its action on p smooth functions $f_1, \dots, f_p \in \mathcal{A}_0$ as follow:

$$P(f_1, \dots, f_p) = \langle P, \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_p \rangle,$$

so P can be regarded as a linear map from $\wedge^p \mathcal{A}_0$ to $\mathcal{A}_0$.

Let $\mathcal{V}^* = \mathrm{Hom}(\wedge^* \mathcal{A}_0, \mathcal{A}_0)$, whose elements are called generalized polyvectors. In particular, we have $\mathcal{V}^0 = \Lambda^0 = \mathcal{A}_0$, and $\mathcal{V}^{<0} = 0$. We regard Λ^* as a subspace of $\mathcal{V}^*$, and it is easy to see that $P \in \mathcal{V}^p$ belongs to Λ^p if and only if

$$P(f \cdot g, f_2, \dots, f_p) = f \cdot P(g, f_2, \dots, f_p) + P(f, f_2, \dots, f_p) \cdot g \tag{1.7}$$

for all $f, g, f_2, \dots, f_p \in \mathcal{A}_0$.

Theorem 1.1 ([29]) *(a) There exists a unique bilinear map $[\,,] : \mathcal{V}^p \times \mathcal{V}^q \to \mathcal{V}^{p+q-1}$ satisfying the following conditions:*

$$[P, f](f_2, \dots, f_p) = P(f, f_2, \dots, f_p), \tag{1.8}$$
$$[P, Q] = (-1)^{pq}[Q, P], \tag{1.9}$$
$$[[P, Q], f] + (-1)^{qp}[[Q, f], P] + [[f, P], Q] = 0, \tag{1.10}$$

where $P \in \mathcal{V}^p$, $Q \in \mathcal{V}^q$, and $f, f_2, \dots, f_p \in \mathcal{A}_0$. It is called the Nijenhuis-Richardson bracket of the generalized polyvectors.

(b) The Nijenhuis-Richardson bracket satisfies the following graded Jacobi identity:

$$(-1)^{pr}[[P, Q], R] + (-1)^{qp}[[Q, R], P] + (-1)^{rq}[[R, P], Q] = 0, \tag{1.11}$$

where $P \in \mathcal{V}^p$, $Q \in \mathcal{V}^q$, and $R \in \mathcal{V}^r$.

Proof: (a) We prove uniqueness first. Let $P \in \mathcal{V}^p$, $Q \in \mathcal{V}^q$. When $p = q = 0$, $[P, Q]$ must vanish, since $\mathcal{V}^{-1} = 0$. When $(p, q) = (1, 0)$, then the property (1.8) implies that $[P, Q] = P(Q)$. The $(0, 1)$ case is similar, due to the property (1.9). When $(p, q) = (1, 1)$, take an $f \in \mathcal{A}_0$, then we have

$$\begin{aligned} &[P, Q](f) = [[P, Q], f] = [[Q, f], P] - [[f, P], Q] \\ =&P(Q(f)) - Q(P(f)). \end{aligned}$$

In general, take $f, f_2, \dots, f_{p+q-1}$, we have

$$\begin{aligned} &[P, Q](f, f_2, \dots, f_{p+q-1}) = [[P, Q], f](f_2, \dots, f_{p+q-1}) \\ =& -\left((-1)^{qp}[[Q, f], P] + [[f, P], Q]\right)(f_2, \dots, f_{p+q-1}), \end{aligned}$$

so the bracket defined on $\mathcal{V}^p \times \mathcal{V}^q$ is determined by the brackets defined on $\mathcal{V}^{p-1} \times \mathcal{V}^q$ and $\mathcal{V}^p \times \mathcal{V}^{q-1}$. Since we have shown the uniqueness for the $0 \le p, q \le 1$ cases, it also holds true for general cases. The uniqueness is proved.

To prove the existence, we recall the product $\bar{\wedge} : \mathcal{V}^p \times \mathcal{V}^q \to \mathcal{V}^{p+q-1}$ defined in [31]:

$$P \bar{\wedge} Q(f_1, \dots, f_{p+q-1}) = \sum_{I \in S_{p,q}} (-1)^{|I|} P(Q(f_{i_1}, \dots, f_{i_q}), f_{i_{q+1}}, \dots, f_{i_{p+q-1}}),$$

where $S_{p,q}$ is the following subset of the symmetry group S_{p+q-1}:

$$S_{p,q} = \left\{ I = (i_1, \dots, i_{p+q-1}) \in S_{p+q-1} \,\middle|\, \begin{array}{c} i_1 < \dots < i_q \\ i_{q+1} < \dots < i_{p+q-1} \end{array} \right\},$$

and $|I|$ is the parity of the permutation I.

The bracket $[\,,]$ can be defined as

$$[P, Q] = (-1)^{(p+1)q} P \bar{\wedge} Q + (-1)^p Q \bar{\wedge} P.$$

We need to show that this bracket satisfies the conditions (1.8)-(1.10). The condition (1.8) and (1.9) are easy to verify. In particular, if $P \in \mathcal{V}^p$, $f, f_2, \dots, f_p \in \mathcal{A}_0$, we have

$$[P, f](f_2, \dots, f_p) = P \bar{\wedge} f(f_2, \dots, f_p) = P(f, f_2, \dots, f_p).$$

We denote $i_f(P) = [P, f]$, then one can show that

$$i_f(P \bar{\wedge} Q) = P \bar{\wedge} i_f(Q) + (-1)^{q+1} i_f(P) \bar{\wedge} Q,$$

which implies the condition (1.10). The existence is proved.

(b) We prove the identity by induction on $p + q + r$. When $r = 0$, it is just the condition (1.10). When $r > 0$, we assume that the identity (1.11) holds true for any p', q', r' satisfying $p' + q' + r' < p + q + r$. Let $P \in \mathcal{V}^p$, $Q \in \mathcal{V}^q$, $R \in \mathcal{V}^r$, and take an $f \in \mathcal{A}_0$, one can show that

$$i_f([[P, Q], R]) = [[i_f(P), Q], R] + (-1)^p [[P, i_f(Q)], R] + (-1)^{p+q} [[P, Q], i_f(R)].$$

Then by using the induction assumption, we obtain

$$i_f\left((-1)^{pr}[[P, Q], R] + (-1)^{qp}[[Q, R], P] + (-1)^{rq}[[R, P], Q]\right) = 0,$$

which implies the identity (1.11). The theorem is proved. □

Remark 1.2 *The above theorem only used the fact that $\mathcal{A}_0$ is a linear space. In next section, we will replace $\mathcal{A}_0$ by another linear space to define the corresponding bracket operation on that space.*

Proposition 1.3 *The Nijenhuis-Richardson bracket can be restricted onto the subspace Λ^*, that is, if $P \in \Lambda^p$, $Q \in \Lambda^q$, then $[P, Q] \in \Lambda^{p+q-1}$. The restricted bracket $[\,,]$ is called the Schouten-Nijenhuis bracket of polyvectors.*

Proof: We prove the proposition by induction on $p + q$.

When $(p,q) = (0,0),(1,0),(0,1),(2,0),(0,2)$, the proposition is trivially true. When $(p,q)=(1,1)$, take $f,g\in\mathcal{A}_0$, we have

$$
\begin{aligned}
&[P,Q](f\cdot g)\\
=&P(Q(f\cdot g))-Q(P(f\cdot g))\\
=&P(f\cdot Q(g)+g\cdot Q(f))-Q(f\cdot P(g)+g\cdot P(f))\\
=&(f\cdot P(Q(g))+P(f)\cdot Q(g)+g\cdot P(Q(f))+P(g)\cdot Q(f))\\
&-(f\cdot Q(P(g))+Q(f)\cdot P(g)+g\cdot Q(P(f))+Q(g)\cdot P(f))\\
=&f\cdot[P,Q](g)+g\cdot[P,Q](f),
\end{aligned}
$$

so $[P,Q]\in\Lambda^1$. From now on we can assume $p+q\geq 3$.

Suppose the proposition holds true for any p',q' satisfying $p'+q'<p+q$, take $f,g,f_2,\ldots,f_{p+q-1}\in\mathcal{A}_0$, we have

$$
\begin{aligned}
&[P,Q](f\cdot g,f_2,\ldots,f_{p+q-1})\\
=&([i_{f_2}(P),Q]+(-1)^p[P,i_{f_2}(Q)])(f\cdot g,f_3,\ldots,f_{p+q-1}).
\end{aligned}
$$

Note that $i_{f_2}(P)\in\Lambda^{p-1}$, $i_{f_2}(Q)\in\Lambda^{q-1}$, so we have

$$
\begin{aligned}
&[i_{f_2}(P),Q](f\cdot g,f_3,\ldots,f_{p+q-1})\\
=&f\cdot[i_{f_2}(P),Q](g,f_3,\ldots,f_{p+q-1})+g\cdot[i_{f_2}(P),Q](f,f_3,\ldots,f_{p+q-1})
\end{aligned}
$$

$$
\begin{aligned}
&[P,i_{f_2}(Q)](f\cdot g,f_3,\ldots,f_{p+q-1})\\
=&f\cdot[P,i_{f_2}(Q)](g,f_3,\ldots,f_{p+q-1})+g\cdot[P,i_{f_2}(Q)](f,f_3,\ldots,f_{p+q-1}),
\end{aligned}
$$

so we have

$$
\begin{aligned}
&[P,Q](f\cdot g,f_2,\ldots,f_{p+q-1})\\
=&f\cdot[P,Q](g,f_2,\ldots,f_{p+q-1}))+g\cdot[P,Q](f,f_2,\ldots,f_{p+q-1})).
\end{aligned}
$$

The proposition is proved. □

Lemma 1.4 *Let $P\in\Lambda^2$ be a bivector, the following conditions are equivalent*

i) P gives the Poisson tensor of a Poisson bracket $\{\,,\}$;

ii) $[P,P]=0$;

iii) The map $d_P:\Lambda^*\to\Lambda^{*+1}$, $Q\mapsto[P,Q]$ satisfies $d_P^2=0$.

Proof: For any $P,Q\in\mathcal{V}^2$ and $f,g,h\in\mathcal{A}_0$, we have

$$
\begin{aligned}
&[P,Q](f,g,h)\\
=&P(Q(f,g),h)+P(Q(g,h),f)+P(Q(h,f),g)\\
&+Q(P(f,g),h)+Q(P(g,h),f)+Q(P(h,f),g).
\end{aligned}
$$

Define $\{f,g\}=P(f,g)$, then we have

$$\{\{f,g\},h\}+\{\{g,h\},f\}+\{\{h,f\},g\}=\frac{1}{2}[P,P](f,g,h).$$

The equivalence of i) and ii) is proved.

For any $Q \in \Lambda^q$, we have

$$[[P,P],Q] + [[P,Q],P] + [[Q,P],P] = 0,$$

which implies that

$$[P,[P,Q]] = -\frac{1}{2}[[P,P],Q].$$

The equivalence of ii) and iii) is proved. □

1.3 Odd-symplectic bracket

The above axiomatic definition of Schouten-Nijenhuis bracket is not very convenient for computation, so we also need another one.

Let $\hat{M} = \Pi(T^*(M))$ be the cotangent bundle of M with fiber's parity reversed, that is, the fiber $T_p^*(M)$ at $\forall p \in M$ is regarded as a super vector space of dimension $(0|n)$. Suppose $(u^1, \ldots, u^n)$ is a set of local coordinates on M, and $(\theta_1, \ldots, \theta_n)$ be the coordinates on fibers with respect to the basis $\mathrm{d}u^1, \ldots, \mathrm{d}u^n$. It is easy to see that, if we change the local coordinate system to another one, say $(\tilde{u}^1, \ldots, \tilde{u}^n)$, the transformation $\theta \mapsto \tilde{\theta}$ is given by the following formula:

$$\tilde{\theta}_\alpha = \frac{\partial u^\beta}{\partial \tilde{u}^\alpha}\theta_\beta, \tag{1.12}$$

which is same with the transformation formula for $\frac{\partial}{\partial u^\alpha}$. Denote by $\hat{\mathcal{A}}_0 = C^\infty(\hat{M})$ the superalgebra of smooth functions on $\hat{M}$.

Lemma 1.5 *There is an isomorphism $\jmath : \hat{\mathcal{A}}_0 \to \Lambda^*$.*

Proof: The superalgebra $\hat{\mathcal{A}}_0$ can be decomposed as

$$\hat{\mathcal{A}}_0 = \bigoplus_{p=0}^{n} \hat{\mathcal{A}}_0^p,$$

where $\hat{\mathcal{A}}_0^p$ is the subspace consisting of functions which have the following form in a local coordinate system:

$$P = P^{\alpha_1\cdots\alpha_p}\theta_{\alpha_1}\cdots\theta_{\alpha_p},$$

where $P^{\alpha_1\cdots\alpha_p}$'s are components of a skew-symmetric tensor of $(p,0)$ type. In particular, $\hat{\mathcal{A}}_0^0 = \mathcal{A}_0$.

We regard Λ^* as the subspace of $\mathcal{V}$ whose elements obey the Leibniz's rule (1.7), and then define the isomorphism $\jmath$ as follow:

$$\jmath : \hat{\mathcal{A}}_0^p \to \Lambda^p, \quad P \mapsto \jmath(P),$$

where the action of $\jmath(P)$ on $f_1, \ldots, f_p \in \mathcal{A}_0$ is given by

$$\jmath(P)(f_1, \ldots, f_p) = \frac{\partial^p P}{\partial\theta_{\alpha_p}\ldots\partial\theta_{\alpha_1}}\frac{\partial f_1}{\partial u^{\alpha_1}}\cdots\frac{\partial f_p}{\partial u^{\alpha_p}}.$$

Then it is not hard to show that $\jmath$ is an isomorphism. □

The cotangent bundle $T^*(M)$ has a canonical symplectic structure, so $\hat{M}$ has a canonical odd-symplectic structure. The corresponding odd-Poisson bracket can be written as

$$[P,Q]_{\hat{\mathcal{A}}_0} = \frac{\partial P}{\partial \theta_\alpha}\frac{\partial Q}{\partial u^\alpha} + (-1)^p \frac{\partial P}{\partial u^\alpha}\frac{\partial Q}{\partial \theta_\alpha}, \tag{1.13}$$

where $P \in \hat{\mathcal{A}}_0^p$, $Q \in \hat{\mathcal{A}}_0^q$. Note that this bracket has other variants (see [24] for example). Here we choose the one that is equivalent to the Schouten-Nijenhuis bracket introduced in the last section.

Proposition 1.6 *We have the following identity:*

$$\jmath([P,Q]_{\hat{\mathcal{A}}_0}) = [\jmath(P), \jmath(Q)]. \tag{1.14}$$

Proof: We only need to show that $[\,,]_{\hat{\mathcal{A}}_0}$ also satisfies the conditions (1.8)-(1.10). This is not a hard task, so we left it to readers. The proposition is proved. □

From now on, we can identify $\hat{\mathcal{A}}_0$ and Λ^*, then write $[\,,]_{\hat{\mathcal{A}}_0}$ as $[\,,]$. A bivector (1.5) can be written as the following form:

$$P = \frac{1}{2} P^{\alpha\beta}\theta_\alpha\theta_\beta.$$

It is a Poisson structure if and only if $[P,P]=0$. Here the bracket $[\,,]$ can be computed by using (1.13).

If $X = X^\alpha \frac{\partial}{\partial u^\alpha}$ is a vector field on M, we can identity it with $X = X^\alpha\theta_\alpha$. Let $H \in \mathcal{A}_0$, the Hamiltonian vector field X_H of H is defined as $X_H = [P,H]$, then we have

$$[X_F, X_G] = X_{\{F,G\}}.$$

In local coordinates, we have

$$X_H = X_H^\beta \theta_\beta, \quad \text{where } X_H^\beta = P^{\alpha\beta}\frac{\partial H}{\partial u^\alpha},$$

so the corresponding ODE can be written as

$$u^\beta_{t_H} = X_H^\beta = \{H, u^\beta\}.$$

2 Infinite dimensional Poisson geometry

2.1 Jet bundles and differential polynomials

In this section, we will define the notion of Hamiltonian structure for an evolutionary partial differential equation of the following form:

$$u_t^\alpha = X^\alpha(u, u', u'', \dots), \quad \alpha = 1, \dots, n, \tag{2.1}$$

where $u^\alpha(x,t)$ are n smooth functions of real variables x and t, and X^α are certain functions of $u = (u^1, \dots, u^n)$, $u' = (u^1_x, \dots, u^n_x)$, …, and so on.

A significant difference between the above equation and usual evolutionary PDE is that it can contain higher derivatives of u^α of any orders, because integrable systems arising from Gromov-Witten theories often take this form. For example, if $X = \mathbb{P}^1$, it is well known that the corresponding integrable system is the Toda lattice hierarchy [35, 23, 32, 16], whose first nontrivial member can be written as

$$u^1_t = \frac{1}{\varepsilon}\left(e^{u^2(x+\varepsilon)} - e^{u^2(x)}\right) = e^{u^2} u^2_x + \sum_{\ell\geq 1} \varepsilon^\ell X^1_\ell(u, u', \dots, u^{(\ell+1)}), \tag{2.2}$$

$$u^2_t = \frac{1}{\varepsilon}\left(u^1(x) - u^1(x-\varepsilon)\right) = u^1_x + \sum_{\ell\geq 1} \varepsilon^\ell X^2_\ell(u, u', \dots, u^{(\ell+1)}). \tag{2.3}$$

Here $\varepsilon = \sqrt{\hbar}$, and X^α_ℓ, which are the Taylor coefficients of the left hand side, are certain polynomials of $u^\alpha_x, \dots, u^\alpha_{(\ell+1)x}$ whose coefficients are smooth functions of u^α. If we introduce the following gradation

$$\deg f(u) = 0, \quad \deg u^\alpha_{\ell\, x} = \ell,$$

then $\deg X^\alpha_\ell = \ell + 1$. To describe functions X^α with these properties, we need to introduce the notion of infinite jet spaces and the algebra of differential polynomials on them.

Let $\hat{N}$ be a super manifold of dimension $(n|m)$. For any integer $k \geq 0$, we define the k-th jet bundle $J^k(\hat{N})$ of $\hat{N}$ as follow: the base manifold of the bundle is $\hat{N}$; the fiber manifold is $\left(\mathbb{R}^{n|m}\right)^k$; the bundle map is denoted by $\pi_{k,0} : J^k(\hat{N}) \to \hat{N}$. Suppose $(z^1, \dots, z^{n+m})$ is a set of coordinates over an open set U of $\hat{N}$, the corresponding coordinates on the fiber are denoted by

$$\{z^{\alpha,s} \mid \alpha = 1, \dots, n+m,\ s = 1, \dots, k\}.$$

In particular, we also take $z^{\alpha,0} = z^\alpha$, then the coordinates for the corresponding open set $\pi^{-1}_{k,0}(U)$ of $J^k(\hat{N})$ can be written as

$$\{z^{\alpha,s} \mid \alpha = 1, \dots, n+m,\ s = 0, \dots, k\}.$$

If we turn to another open set $\tilde{U}$ with coordinates $(\tilde{z}^1, \dots, \tilde{z}^{n+m})$, then the transition functions of the bundle $J^k(\hat{N})$ are given by

$$\begin{aligned}
\tilde{z}^{\alpha,1} =& z^{\beta,1}\frac{\partial \tilde{z}^\alpha}{\partial z^\beta},\\
\tilde{z}^{\alpha,2} =& z^{\beta,2}\frac{\partial \tilde{z}^\alpha}{\partial z^\beta} + z^{\beta_1,1} z^{\beta_2,1}\frac{\partial^2 \tilde{z}^\alpha}{\partial z^{\beta_2}\partial z^{\beta_1}},\\
\tilde{z}^{\alpha,3} =& z^{\beta,3}\frac{\partial \tilde{z}^\alpha}{\partial z^\beta} + 3 z^{\beta_1,2} z^{\beta_2,1}\frac{\partial^2 \tilde{z}^\alpha}{\partial z^{\beta_2}\partial z^{\beta_1}}\\
&+ z^{\beta_1,1} z^{\beta_2,1} z^{\beta_3,1}\frac{\partial^3 \tilde{z}^\alpha}{\partial z^{\beta_3}\partial z^{\beta_2}\partial z^{\beta_1}}, \quad \dots
\end{aligned}$$

The rule for these transition functions is very simple: if $z^{\alpha,s}$ gives the s-th derivative of a curve $\gamma : (-\epsilon, \epsilon) \to \hat{N}$ in the local chart U, then $\tilde{z}^{\alpha,s}$ should be the same derivatives in the local chart $\tilde{U}$. In general, we have

$$\tilde{z}^{\alpha,s+1} = \sum_{t=0}^{s} z^{\beta,t+1}\frac{\partial \tilde{z}^{\alpha,s}}{\partial z^{\beta,t}}. \tag{2.4}$$

Note that these transition functions are not linear in $z^{\alpha,s}$, so jet bundles are not vector bundle, thought their fibers are vector spaces.

Definition 2.1 *(a) A function $f \in C^\infty(J^k(\hat{N}))$ is called a differential polynomial if it is a polynomial of jet variables.*

More precisely, let U be an open set of $\hat{N}$ with coordinates $(z^1, \dots, z^{n+m})$, and $\pi_{k,0}^{-1}(U)$ be the corresponding open set of $J^k(\hat{N})$ with coordinates

$$\{z^{\alpha,s} \mid \alpha = 1, \dots, n+m,\ s = 0, \dots, k\},$$

then we have

$$f|_{\pi_{k,0}^{-1}(U)} \in C^\infty(U)[z^{\alpha,s} \mid \alpha = 1, \dots, n+m,\ s = 1, \dots, k].$$

This definition is independent of the choice of the open set U because of the definition of transition functions (2.4).

All differential polynomials form a subalgebra of $C^\infty(J^k(\hat{N}))$. We denote this subalgebra by ${}^\circ\hat{\mathcal{A}}^{(k)}(\hat{N})$.

(b) We define

$$\deg f(z) = 0 \text{ if } f(z) \in C^\infty(\hat{N}), \quad \deg z^{\alpha,s} = s \text{ if } s \geq 1,$$

and extend it to the whole ${}^\circ\hat{\mathcal{A}}^{(k)}(\hat{N})$, then ${}^\circ\hat{\mathcal{A}}^{(k)}(\hat{N})$ becomes a graded ring.

For any $f \in {}^\circ\hat{\mathcal{A}}^{(k)}(\hat{N})$, we can uniquely decompose it as follow

$$f = f_{d_{\min}} + f_{d_{\min}+1} + \cdots + f_{d_{\max}},$$

where $f_{d_{\min}}, f_{d_{\max}} \neq 0$ and $\deg f_d = d$. The number $d_{\min}$ is called the valuation of f, which is denoted by $\nu(f)$. (The number $d_{\max}$ can be called the degree of f, but we never use this notion.)

(c) We define a distance function over ${}^\circ\hat{\mathcal{A}}^{(k)}(\hat{N})$:

$$dist(f, g) = e^{-\nu(f-g)}, \quad \forall f, g \in {}^\circ\hat{\mathcal{A}}^{(k)}(\hat{N}).$$

Then denote by $\hat{\mathcal{A}}^{(k)}(\hat{N})$ the completion of ${}^\circ\hat{\mathcal{A}}^{(k)}(\hat{N})$ with respect to dist.

More precisely, let $f \in \hat{\mathcal{A}}^{(k)}(\hat{N})$, and U be an open set of $\hat{N}$, then we have

$$f|_{\pi_{k,0}^{-1}(U)} \in C^\infty(U)[[z^{\alpha,s} \mid \alpha = 1, \dots, n+m,\ s = 1, \dots, k]].$$

Here the formal power series ring $C^\infty(U)[[z^{\alpha,s}]]$ is completed by using the distance function dist.

We are only interested in $\hat{\mathcal{A}}^{(k)}(\hat{N})$, and will never use the notation ${}^\circ\hat{\mathcal{A}}^{(k)}(\hat{N})$ and the distance function *dist*. So, to abuse of language, we will call elements of $\hat{\mathcal{A}}^{(k)}(\hat{N})$ differential polynomials from now on, though they are actually formal power series in general. To indicate the degrees of every homogeneous components, we may write $f \in \hat{\mathcal{A}}^{(k)}(\hat{N})$ as

$$f = f_0 + f_1 + f_2 + \cdots = f_0 + \varepsilon\, f_1 + \varepsilon^2\, f_2 + \cdots, \quad \text{where } \deg f_d = d.$$

Then the topology on $\hat{\mathcal{A}}^{(k)}(\hat{N})$ is just the ε-adic topology.

For $k \geq l \geq 0$, there is a projection map $\pi_{k,l} : J^k(\hat{N}) \to J^l(\hat{N})$, which just forgets the coordinates $z^{\alpha,s}$ with $s > l$. Jet bundles and the projection maps among them form an inverse system

$$\left(\{J^k(\hat{N})\}_{k\geq 0},\ \{\pi_{k,l}\}_{k\geq l\geq 0}\right).$$

We denote the inverse limit of this inverse system by $J^\infty(\hat{N})$, and name it the infinite jet space of $\hat{N}$.

The projection $\pi_{k,l}$ ($k \geq l$) induces a pullback map $\pi^*_{k,l} : \hat{\mathcal{A}}^{(l)}(\hat{N}) \to \hat{\mathcal{A}}^{(k)}(\hat{N})$. The differential polynomial algebras and the pullback maps among them form a direct system

$$\left(\{\hat{\mathcal{A}}^{(k)}(\hat{N})\}_{k\geq 0},\ \{\pi^*_{k,l}\}_{k\geq l\geq 0}\right).$$

We denote the direct limit of this direct system by $\hat{\mathcal{A}}(\hat{N})$, and name it the differential polynomial ring of $\hat{N}$.

Note that the maps $\pi^*_{k,l}$ are all injective, so every $\hat{\mathcal{A}}^{(k)}(\hat{N})$ can be regarded as a subalgebra of $\hat{\mathcal{A}}(\hat{N})$. These subalgebras define a filtration on $\hat{\mathcal{A}}(\hat{N})$:

$$\hat{\mathcal{A}}^{(0)}(\hat{N}) \subset \hat{\mathcal{A}}^{(1)}(\hat{N}) \subset \hat{\mathcal{A}}^{(2)}(\hat{N}) \subset \cdots \subset \hat{\mathcal{A}}(\hat{N}).$$

The maps $\pi^*_{k,l}$ preserve the gradation on $\hat{\mathcal{A}}^{(k)}(\hat{N})$, so $\hat{\mathcal{A}}(\hat{N})$ also has a gradation

$$\hat{\mathcal{A}}(\hat{N}) = \bigoplus_{d\geq 0} \hat{\mathcal{A}}_d(\hat{N}), \quad \hat{\mathcal{A}}_d(\hat{N}) = \{f \in \hat{\mathcal{A}}^{(d)}(\hat{N}) |\, \deg f = d\},$$

which is called the standard gradation of $\hat{\mathcal{A}}(\hat{N})$. In particular, $\hat{\mathcal{A}}_0(\hat{N}) = C^\infty(\hat{N})$.

Let M be a smooth manifold of dimension n, and $\hat{M} = \Pi\,(T^*(M))$ be the odd-symplectic cotangent bundle introduced in the last section. We can define $J^\infty(M)$ and $J^\infty(\hat{M})$ as above, whose differential polynomial algebras are denoted by $\mathcal{A} = \hat{\mathcal{A}}(M)$ and $\hat{\mathcal{A}} = \hat{\mathcal{A}}(\hat{M})$ respectively. Their local coordinates are written as

$$\{u^{\alpha,s} \mid \alpha = 1, \dots, n,\ s \geq 0\}$$

and

$$\{u^{\alpha,s},\ \theta^s_\alpha \mid \alpha = 1, \dots, n,\ s \geq 0\}$$

respectively. The algebra $\mathcal{A}$ can be identified with the subalgebra of $\hat{\mathcal{A}}$ whose elements don't depend on any θ^s_α. The superalgebra $\hat{\mathcal{A}}$ has another gradation

$$\hat{\mathcal{A}} = \bigoplus_{p\geq 0} \hat{\mathcal{A}}^p, \quad \hat{\mathcal{A}}^p = \{f = \sum_{s_1,\dots,s_p\geq 0} f^{\alpha_1,\dots,\alpha_p}_{s_1,\dots,s_p} \theta^{s_1}_{\alpha_1} \cdots \theta^{s_p}_{\alpha_p} \mid f^{\alpha_1,\dots,\alpha_p}_{s_1,\dots,s_p} \in \mathcal{A}\},$$

which is called the super gradation of $\hat{\mathcal{A}}$. We also use the notation $\hat{\mathcal{A}}^p_d = \hat{\mathcal{A}}^p \cap \hat{\mathcal{A}}_d$. In particular, we have $\hat{\mathcal{A}}^0 = \mathcal{A}$, $\hat{\mathcal{A}}^0_0 = \mathcal{A}_0 = C^\infty(M)$. This explains the notations we used in the last section.

2.2 Evolutionary partial differential equations

We can define evolutionary PDEs of the form (2.1) now. Let us prove two lemmas first.

Lemma 2.2 *The following operator*

$$\partial_{\hat{N}} = \sum_{s\geq 0} z^{\alpha,s+1} \frac{\partial}{\partial z^{\alpha,s}}$$

defines a global vector field on $J^\infty(\hat{N})$, and it also defines a derivation of $\hat{\mathcal{A}}(\hat{N})$.

Proof: According to the definition (2.4) of transition functions of the bundle $J^\infty(\hat{N})$, we have

$$\partial_{\hat{N}} = \sum_{s\geq 0} z^{\alpha,s+1}\frac{\partial}{\partial z^{\alpha,s}} = \sum_{s\geq 0} \tilde{z}^{\alpha,s+1}\frac{\partial}{\partial \tilde{z}^{\alpha,s}}.$$

The lemma is proved. □

When $\hat{N} = \hat{M}$ (or M), $\partial_{\hat{N}}$ has the following expression

$$\partial_{\hat{M}} = \sum_{s\geq 0}\left(u^{\alpha,s+1}\frac{\partial}{\partial u^{\alpha,s}} + \theta_\alpha^{s+1}\frac{\partial}{\partial \theta_\alpha^s}\right) \quad \left(\text{or } \partial_M = \sum_{s\geq 0}\left(u^{\alpha,s+1}\frac{\partial}{\partial u^{\alpha,s}}\right)\right),$$

Note that $\mathcal{A} = \hat{\mathcal{A}}^0$, and $\partial_M = \partial_{\hat{M}}|_{\hat{\mathcal{A}}^0}$, so we denote them by $\partial = \partial_{\hat{M}} = \partial_M$ to abuse of notation.

Lemma 2.3 *Let $X : \hat{\mathcal{A}}(\hat{N}) \to \hat{\mathcal{A}}(\hat{N})$ be a continuous derivation such that $[X, \partial] = 0$, then we have*

$$X = \sum_{s\geq 0} \partial^s(X^\alpha)\frac{\partial}{\partial z^{\alpha,s}}, \tag{2.5}$$

where $X^\alpha \in \hat{\mathcal{A}}(\hat{N})$.

Proof: Without loss of generality, we can assume that X is homogeneous with respect to the super gradation of $\hat{\mathcal{A}}(\hat{N})$, that is $X(\hat{\mathcal{A}}^p(\hat{N})) \subset \hat{\mathcal{A}}^{p+|X|}(\hat{N})$, where $|X| \in \mathbb{Z}$ is called the super degree of X. Then a derivation is a linear map $X : \hat{\mathcal{A}}(\hat{N}) \to \hat{\mathcal{A}}(\hat{N})$ such that

$$X(f\cdot g) = X(f)\cdot g + (-1)^{|X|\,|f|} f\cdot X(g),$$

where $f \in \hat{\mathcal{A}}^{|f|}(\hat{N})$, $g \in \hat{\mathcal{A}}(\hat{N})$.

If $f \in \hat{\mathcal{A}}^{(n)}(\hat{N})$ for some $n \in \mathbb{N}$, then it is easy to see that

$$X(f) = \sum_{s=0}^{n} X(z^{\alpha,s})\frac{\partial f}{\partial z^{\alpha,s}} = \sum_{s\geq 0}\partial^s(X^\alpha)\frac{\partial f}{\partial z^{\alpha,s}},$$

where $X^\alpha = X(z^\alpha) \in \hat{\mathcal{A}}(\hat{N})$.

If f doesn't belong to any $\hat{\mathcal{A}}^{(n)}(\hat{N})$,

$$f = \sum_{d\geq 0} f_d, \quad \text{where } f_d \in \hat{\mathcal{A}}_d \subset \hat{\mathcal{A}}^{(d)}(\hat{N}),$$

then we have

$$\begin{aligned}
X(f) &= X\left(\lim_{n\to\infty}\sum_{d=0}^{n} f_d\right)\\
&= \lim_{n\to\infty} X\left(\sum_{d=0}^{n} f_d\right) \quad (\Leftarrow X \text{ is continuous})\\
&= \lim_{n\to\infty}\left(\sum_{s\geq 0}\partial^s(X^\alpha)\frac{\partial}{\partial z^{\alpha,s}}\right)\left(\sum_{d=0}^{n} f_d\right) \quad \left(\Leftarrow \sum_{d=0}^{n} f_d \in \hat{\mathcal{A}}^{(n)}\right)\\
&= \sum_{s\geq 0}\partial^s(X^\alpha)\frac{\partial f}{\partial z^{\alpha,s}}.
\end{aligned}$$

The last equality holds true because $\partial^p(X^\alpha)\frac{\partial}{\partial z^{\alpha,s}} : \hat{\mathcal{A}}(\hat{N}) \to \hat{\mathcal{A}}(\hat{N})$ is continuous for all p, and the summation $\sum_{p\geq 0}\partial^p(X^\alpha)\frac{\partial}{\partial z^{\alpha,s}}$ is uniformly convergent, so the summation itself is also continuous. □

If we have an evolutionary PDE (2.1) with $X^\alpha \in \mathcal{A}$, then for any $f \in \mathcal{A}$, we have

$$f_t = \sum_{s\geq 0} (u^{\alpha,s})_t \frac{\partial f}{\partial u^{\alpha,s}} = \sum_{s\geq 0} \partial^s (X^\alpha) \frac{\partial f}{\partial u^{\alpha,s}},$$

which is just $X(f)$ with X given by (2.5), so we have the following definition.

Definition 2.4 *(a) We denote by* $\mathrm{Der}(\hat{N})$ *the Lie algebra of continuous derivations over* $\hat{\mathcal{A}}(\hat{N})$*, and define*

$$\hat{\mathcal{E}}(\hat{N}) = \mathrm{Der}(\hat{N})^\partial = \{X \in \mathrm{Der}(\hat{N}) \mid [X, \partial] = 0\},$$

whose elements are called evolutionary vector field on $J^\infty(\hat{N})$*.*

(b) According to Lemma 2.3, an element $X \in \hat{\mathcal{E}}(\hat{N})$ *always takes the following form:*

$$X = \sum_{s\geq 0} \partial^s(X^\alpha)\frac{\partial}{\partial z^{\alpha,s}}.$$

We denote it by $X = (X^\alpha)$ *for short. The differential polynomials* X^α*'s are called the components of* X*.*

(c) We denote $\mathcal{E} = \hat{\mathcal{E}}(M)$ *and* $\hat{\mathcal{E}} = \hat{\mathcal{E}}(\hat{M})$*.*

It is easy to see that $\mathcal{E}$ is a Lie algebra, and $\hat{\mathcal{E}}$ is a graded Lie algebra.

2.3 Conserved Quantities

To develop a Hamiltonian formalism for the equation (2.1), we still need the notion of *conserved quantity*. Roughly speaking, a conserved quantity for (2.1) is a functional

$$I[u] = \int_{\mathbb{R}} f(u, u', u'', \ldots, u^{(N)})\mathrm{d}x$$

such that if $u(x,t)$ is a solution for (2.1), then

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \int_{\mathbb{R}} f_t\,\mathrm{d}x = \int_{\mathbb{R}} X(f)\,\mathrm{d}x = 0.$$

This definition is not very convenient, because we need some conditions on u and f to ensure that the integrations are convergent. A better choice is to replace $\mathbb{R}$ by $S^1 = \mathbb{R}/\mathbb{Z}$, and assume that $u(x) = (u^1(x), \ldots, u^n(x))$ is actually the coordinates of a smooth map $\phi : S^1 \to M$.

Let $\mathcal{L}(M) = C^\infty(S^1, M)$ be the loop space of M. For any $\phi \in \mathcal{L}(M)$, we can lift it to a map $\phi^k : S^1 \to J^k(M)$ for all $k = 1, 2, \ldots, \infty$. Then for any $f \in C^\infty(J^\infty(M))$, we can define a smooth function $(\phi^\infty)^*(f) : S^1 \to \mathbb{R}$, $x \mapsto f(\phi^\infty(x))$, and then define the following functional:

$$I_f[\phi] = \int_{S_1} (\phi^\infty)^*(f)(x)\mathrm{d}x.$$

Lemma 2.5 *Let* $\mathring{\mathcal{F}}$ *be the linear space of functionals of the form* I_f*, and* $I : C^\infty(J^\infty(M)) \to \mathring{\mathcal{F}}$ *be the map* $f \mapsto I_f$*. Then* $\mathrm{Ker}(I) = \mathrm{Im}(\partial)$*, hence we have an isomorphism* $\mathring{\mathcal{F}} \cong C^\infty(J^\infty(M))/\partial(C^\infty(J^\infty(M)))$*.*

Proof: By definition,

$$C^\infty(J^\infty(M)) = \varinjlim_k C^\infty(J^k(M)),$$

if $f \in C^\infty(J^\infty(M))$, then there exists $k \in \mathbb{N}$ such that $f \in C^\infty(J^k(M))$.

If $f \in \mathrm{Im}(\partial)$, there exists $g \in C^\infty(J^k(M))$ such that $f = \partial g$, so we have

$$I_f[\phi] = \int_{S^1} (\partial g)(\phi^\infty(x))\mathrm{d}x = g(\phi^\infty(x))|_0^1 = 0,$$

that is, $\mathrm{Im}(\partial) \subset \mathrm{Ker}(I)$.

Conversely, if $I_f[\phi] = 0$ for any $\phi \in \mathcal{L}(M)$, we need to construct $g \in C^\infty(J^k(M))$ such that $f = \partial g$.

Suppose M is connected (otherwise, we can do the following for each of M's connected component), fix a point $P_0 \in J^k(M)$, and take $Q_0 = \pi_{k,0}(P_0) \in M$. For any $P \in J^k(M)$, let $Q = \pi_{k,0}(P) \in M$. There exists a path $\gamma : [0, 1/2] \to M$ such that

$$\gamma(0) = Q_0, \quad \gamma(1/2) = Q, \quad \gamma^k(0) = P_0, \quad \gamma^k(1/2) = P,$$

where $\gamma^k : [0, 1/2] \to J^k(M)$ is the lifted map. Then define

$$g : J^k(M) \to \mathbb{R}, \quad P \mapsto g(P) = \int_0^{1/2} f(\gamma^k(x))\mathrm{d}x.$$

This definition is independent of the choice of γ (because $f \in \mathrm{Ker}(I)$), and it is easy to see that $\partial g = f$. The lemma is proved. □

The above lemma shows that, even the loop space is not necessary: we can define $\mathring{\mathcal{F}}$ as the cokernel of ∂. Inspired by this fact, we give the following definition.

Definition 2.6 *(a) We define* $\hat{\mathcal{F}}(\hat{N}) = \hat{\mathcal{A}}(\hat{N})/\partial\hat{\mathcal{A}}(\hat{N})$*, whose elements are called local functionals on* $\hat{N}$*. A local functional* $f + \partial\hat{\mathcal{A}}(\hat{N})$ *is usually denoted by* $\int f\mathrm{d}x$*, and the representative* f *is called a density of this functional.*

(b) The space $\hat{\mathcal{F}}(\hat{N})$ *has an natural* $\hat{\mathcal{E}}(\hat{N})$*-module structure,*

$$\left(X, F = \int f\mathrm{d}x\right) \mapsto X(F) = \int X(f)\mathrm{d}x.$$

A local functional $F \in \hat{\mathcal{F}}(\hat{N})$ *is called a conserved quantity of* $X \in \hat{\mathcal{E}}(\hat{N})$*, if* $X(F) = 0$*.*

(c) We denote $\mathcal{F} = \hat{\mathcal{F}}(M)$ *and* $\hat{\mathcal{F}} = \hat{\mathcal{F}}(\hat{M})$*. Note that* ∂ *preserves the two gradations on* $\hat{\mathcal{A}}$*, so there are induced standard gradation and super gradation on* $\hat{\mathcal{F}}$*. We denote them by*

$$\hat{\mathcal{F}} = \bigoplus_{d\geq 0} \hat{\mathcal{F}}_d = \bigoplus_{p\geq 0} \hat{\mathcal{F}}^p,$$

and $\hat{\mathcal{F}}_d^p = \hat{\mathcal{F}}_d \cap \hat{\mathcal{F}}^p$*. In particular,* $\mathcal{F} = \hat{\mathcal{F}}^0$*.*

Lemma 2.7 *Let $X = (X^\alpha) \in \hat{\mathcal{E}}(\hat{N})$, $F = \int f \mathrm{d}x \in \hat{\mathcal{F}}(\hat{N})$, then we have*

$$X(F) = \int \left(X^\alpha \frac{\delta F}{\delta z^\alpha} \right) \mathrm{d}x,$$

where

$$\frac{\delta F}{\delta z^\alpha} = \sum_{s \geq 0} (-\partial)^s \frac{\partial f}{\partial z^{\alpha,s}}$$

is the variational derivative of F with respect to z^α.

Proof: In the space $\hat{\mathcal{F}}(\hat{N})$, we still have integration by parts, so

$$\begin{aligned} X(F) &= \int \left(\sum_{s \geq 0} \partial^s (X^\alpha) \frac{\partial f}{\partial z^{\alpha,s}} \right) \mathrm{d}x \\ &= \int \left(X^\alpha \sum_{s \geq 0} (-\partial)^s \frac{\partial f}{\partial z^{\alpha,s}} \right) \mathrm{d}x. \end{aligned}$$

The lemma is proved. □

2.4 Hamiltonian structures

We are ready to define Hamiltonian structures for the evolutionary PDE (2.1). Similar to the finite-dimensional case, a Hamiltonian structure on M is a Lie bracket over the space of local functionals (i.e. $\mathcal{F}$) whose action is given by certain differential operations in a local chart.

Definition 2.8 *(a) Let $\mathcal{V}^* = \mathrm{Hom}(\wedge^* \mathcal{F}, \mathcal{F})$, whose elements are called generalized variational polyvector. According to Theorem 1.1, there is a unique bracket operation $[\,,] : \mathcal{V}^p \times \mathcal{V}^q \to \mathcal{V}^{p+q-1}$ satisfying the condition* (1.8)-(1.10) *with $\mathcal{A}_0$ replaced by $\mathcal{F}$ and the condition* (1.11). *We still call it the Nijenhuis-Richardson bracket.*

(b) A generalized variational p-vector $P \in \mathcal{V}^p$ is called a variational p-vector, if its action on $F_1, \dots, F_p \in \mathcal{F}$ is given by

$$P(F_1, \dots, F_p) = \int \left(\sum_{s_1, \dots, s_p \geq 0} P^{\alpha_1, \dots, \alpha_p}_{s_1, \dots, s_p} \partial^{s_1} \left(\frac{\delta F_1}{\delta u^{\alpha_1}} \right) \cdots \partial^{s_p} \left(\frac{\delta F_p}{\delta u^{\alpha_p}} \right) \right) \mathrm{d}x, \tag{2.6}$$

where $P^{\alpha_1, \dots, \alpha_p}_{s_1, \dots, s_p} \in \mathcal{A}$. The space of variational p-vectors is denoted by Λ^p. We denote by $\Lambda^* = \bigoplus_{p \geq 0} \Lambda^p$, which is a subspace of $\mathcal{V}^*$.

(c) A variational bivector $P \in \Lambda^2$ is called a Hamiltonian structure, if $[P, P] = 0$.

We have an infinite-dimensional analogue of Proposition 1.3.

Proposition 2.9 *If $P \in \Lambda^p$, $Q \in \Lambda^q$, then $[P, Q] \in \Lambda^{p+q-1}$.*

The definition (2.6) of variational polyvectors is very complicated. It is not easy to determine whether a generalized variational polyvector $P \in \mathcal{V}^*$ belongs to Λ^*, so we cannot prove the above proposition directly. In what follows, we will give another description of Λ^*, then prove the proposition by using the odd-symplectic bracket on $\hat{\mathcal{F}}$.

Lemma 2.10 *Define a map* $\jmath : \hat{\mathcal{F}}^p \to \Lambda^p$,

$$\begin{aligned} P = \int \tilde{P}\mathrm{d}x \mapsto & \jmath(P)(F_1, \dots, F_p) \\ = & \int \left(\sum_{s_1,\dots,s_p \ge 0} \frac{\partial^p \tilde{P}}{\partial \theta_{\alpha_p}^{s_p} \cdots \partial \theta_{\alpha_1}^{s_1}} \partial^{s_1} \left(\frac{\delta F_1}{\delta u^{\alpha_1}} \right) \cdots \partial^{s_p} \left(\frac{\delta F_p}{\delta u^{\alpha_p}} \right) \right) \mathrm{d}x. \end{aligned}$$

Then $\jmath(P)$ *is independent of the choice of the density* $\tilde{P}$, *and* $\jmath$ *is surjective with* $\mathrm{Ker}(\jmath) = \mathbb{R}\omega \subset \hat{\mathcal{F}}^1$, *where* $\omega = \int (u^{\alpha,1}\theta_\alpha)\mathrm{d}x$. *So we have the isomorphisms* $\Lambda^p \cong \hat{\mathcal{F}}^p$ $(p \neq 1)$, *and* $\Lambda^1 \cong \hat{\mathcal{F}}^1/\mathbb{R}\omega$.

We have to omit the proof of this lemma because of its length. In [29], we proved a generalization of this lemma in §2.3. One can easily reduce that proof to the present case.

Define the action of $P \in \hat{\mathcal{F}}^p$ on $F_1, \dots, F_p \in \mathcal{F}$ by

$$P(F_1, \dots, F_p) = \jmath(P)(F_1, \dots, F_p).$$

Then we have the following lemma.

Lemma 2.11 *For* $P \in \hat{\mathcal{F}}^p$, $Q \in \hat{\mathcal{F}}^q$, *define*

$$[P, Q] = \int \left(\frac{\delta P}{\delta \theta_\alpha} \frac{\delta Q}{\delta u^\alpha} + (-1)^p \frac{\delta P}{\delta u^\alpha} \frac{\delta Q}{\delta \theta_\alpha} \right) \mathrm{d}x,$$

then the operation $[\,,]$ *satisfies the condition* (1.8)-(1.10) *with* $\mathcal{A}_0$ *replaced by* $\mathcal{F}$ *and the condition* (1.11), *hence* $(\hat{\mathcal{F}}, [\,,])$ *forms a graded Lie algebra. In particular, its center is given by* $\mathbb{R}\omega$.

Proof: Suppose $P = \int \tilde{P}\mathrm{d}x \in \hat{\mathcal{F}}^p$, $F \in \mathcal{F}$, then

$$[P, F] = \int \left(\frac{\delta P}{\delta \theta_\alpha} \frac{\delta F}{\delta u^\alpha} \right) \mathrm{d}x = \int \left(\sum_{s \ge 0} \frac{\partial \tilde{P}}{\partial \theta_\alpha^s} \partial^s \left(\frac{\delta F}{\delta u^\alpha} \right) \right) \mathrm{d}x,$$

so we have

$$\begin{aligned} & [P, F](F_2, \dots, F_p) \\ = & \int \left(\sum_{s_2,\dots,s_p \ge 0} \frac{\partial^p}{\partial \theta_{\alpha_p}^{s_p} \cdots \partial \theta_{\alpha_2}^{s_2}} \left(\sum_{s_1 \ge 0} \frac{\partial \tilde{P}}{\partial \theta_{\alpha_1}^{s_1}} \partial^{s_1} \left(\frac{\delta F}{\delta u^{\alpha_1}} \right) \right) \right. \\ & \left. \partial^{s_2} \left(\frac{\delta F_2}{\delta u^{\alpha_2}} \right) \cdots \partial^{s_p} \left(\frac{\delta F_p}{\delta u^{\alpha_p}} \right) \right) \mathrm{d}x \\ = & \int \left(\sum_{s_1,\dots,s_p \ge 0} \frac{\partial^p \tilde{P}}{\partial \theta_{\alpha_p}^{s_p} \cdots \partial \theta_{\alpha_1}^{s_1}} \partial^{s_1} \left(\frac{\delta F_1}{\delta u^{\alpha_1}} \right) \cdots \partial^{s_p} \left(\frac{\delta F_p}{\delta u^{\alpha_p}} \right) \right) \mathrm{d}x \\ = & P(F, F_2, \dots, F_p). \end{aligned}$$

The identity (1.8) is proved.

Suppose $P \in \hat{\mathcal{F}}^p$, $Q \in \hat{\mathcal{F}}^q$, then we have

$$\begin{aligned}
[P,Q] =& \int \left(\frac{\delta P}{\delta \theta_\alpha} \frac{\delta Q}{\delta u^\alpha} + (-1)^p \frac{\delta P}{\delta u^\alpha} \frac{\delta Q}{\delta \theta_\alpha} \right) \mathrm{d}x \\
=& \int \left((-1)^{(p-1)q} \frac{\delta Q}{\delta u^\alpha} \frac{\delta P}{\delta \theta_\alpha} + (-1)^{p+p(q-1)} \frac{\delta Q}{\delta \theta_\alpha} \frac{\delta P}{\delta u^\alpha} \right) \mathrm{d}x \\
=&(-1)^{pq} \int \left(\frac{\delta Q}{\delta \theta_\alpha} \frac{\delta P}{\delta u^\alpha} + (-1)^q \frac{\delta Q}{\delta u^\alpha} \frac{\delta P}{\delta \theta_\alpha} \right) \mathrm{d}x \\
=&(-1)^{pq}[Q,P].
\end{aligned}$$

The identity (1.9) is proved.

The identity (1.10) is a special case of (1.11), so we only need to prove the latter one. For any $P \in \hat{\mathcal{F}}^p$, we define an operator $D_P : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$

$$D_P = \sum_{s \geq 0} \left(\partial^s \left(\frac{\delta P}{\delta \theta_\alpha} \right) \frac{\partial}{\partial u^{\alpha,s}} + (-1)^p \partial^s \left(\frac{\delta P}{\delta u^\alpha} \right) \frac{\partial}{\partial \theta_\alpha^s} \right), \tag{2.7}$$

then it is easy to see that $D_P(\hat{\mathcal{A}}^q) \subset \hat{\mathcal{A}}^{p+q-1}$, $[D_P, \partial] = 0$, and $[P,Q] = \int D_P(Q)\mathrm{d}x$ for any $Q \in \hat{\mathcal{F}}^q$. The identity (1.11) is equivalent to the following identity:

$$(-1)^{p-1} D_{[P,Q]} = D_P \circ D_Q - (-1)^{(p-1)(q-1)} D_Q \circ D_P, \tag{2.8}$$

which is a corollary of the following identity:

$$\frac{\delta}{\delta u^\alpha}[P,Q] = D_P \left(\frac{\delta Q}{\delta u^\alpha} \right) + (-1)^{pq} D_Q \left(\frac{\delta P}{\delta u^\alpha} \right), \tag{2.9}$$

$$(-1)^{p-1} \frac{\delta}{\delta \theta_\alpha}[P,Q] = D_P \left(\frac{\delta Q}{\delta \theta_\alpha} \right) - (-1)^{(p-1)(q-1)} D_Q \left(\frac{\delta P}{\delta \theta_\alpha} \right). \tag{2.10}$$

To prove the identity (2.9), (2.10), we introduce the following operators

$$\delta_{\alpha,s} = \sum_{t \geq 0} (-1)^t \binom{t+s}{s} \partial^t \frac{\partial}{\partial u^{\alpha,s}},$$

$$\delta_s^\alpha = \sum_{t \geq 0} (-1)^t \binom{t+s}{s} \partial^t \frac{\partial}{\partial \theta_\alpha^s},$$

which are called the higher Euler operators. In particular,

$$\delta_{\alpha,0} = \frac{\delta}{\delta u^\alpha}, \quad \delta_0^\alpha = \frac{\delta}{\delta \theta_\alpha},$$

and they satisfy the following identities:

$$\delta_{\alpha,0}(f \cdot g) = \sum_{t \geq 0} (-1)^t \left(\delta_{\alpha,t}(f) \partial^t(g) + \partial^t(f) \delta_{\alpha,t}(g) \right),$$

$$\delta_{\alpha,t} \delta_{\beta,0} = (-1)^t \frac{\partial}{\partial u^{\beta,t}} \delta_{\alpha,0}, \quad \delta_{\alpha,t} \delta_0^\beta = (-1)^t \frac{\partial}{\partial \theta_\beta^t} \delta_{\alpha,0}.$$

Then we have

$$
\begin{aligned}
&\delta_{\alpha,0}[P,Q]\\
=&\delta_{\alpha,0}\left(\delta_0^\beta(P)\delta_{\beta,0}(Q)+(-1)^p\delta_{\beta,0}(P)\delta_0^\beta(Q)\right)\\
=&\sum_{t\geq 0}(-1)^t\left(\delta_{\alpha,t}(\delta_0^\beta(P))\partial^t(\delta_{\beta,0}(Q))+\partial^t(\delta_0^\beta(P))\delta_{\alpha,t}(\delta_{\beta,0}(Q))\right)\\
&+(-1)^p\sum_{t\geq 0}(-1)^t\left(\delta_{\alpha,t}(\delta_{\beta,0}(P))\partial^t(\delta_0^\beta(Q))+\partial^t(\delta_{\beta,0}(P))\delta_{\alpha,t}(\delta_0^\beta(Q))\right)\\
=&\sum_{t\geq 0}\left(\frac{\partial(\delta_{\alpha,0}(P))}{\partial\theta_\beta^t}\partial^t(\delta_{\beta,0}(Q))+\partial^t(\delta_0^\beta(P))\frac{\partial(\delta_{\alpha,0}(Q))}{\partial u^{\beta,t}}\right.\\
&\left.+(-1)^p\left(\frac{\partial(\delta_{\alpha,0}(P))}{\partial u^{\beta,t}}\partial^t(\delta_0^\beta(Q))+\partial^t(\delta_{\beta,0}(P))\frac{\partial(\delta_{\alpha,0}(Q))}{\partial\theta_\beta^t}\right)\right)\\
=&D_P(\delta_{\alpha,0}(Q))+(-1)^{pq}D_Q(\delta_{\alpha,0}(P)).
\end{aligned}
$$

The identity (2.9) is proved. The identity (2.10) can be proved similarly.

Suppose $Q=\int\tilde{Q}\mathrm{d}x\in\hat{\mathcal{F}}^q$, we have

$$
\begin{aligned}
[\omega,Q]&=\int\left(\frac{\delta\omega}{\delta\theta_\alpha}\frac{\delta Q}{\delta u^\alpha}-\frac{\delta\omega}{\delta u^\alpha}\frac{\delta Q}{\delta\theta_\alpha}\right)\mathrm{d}x\\
&=\int\left(u^{\alpha,1}\frac{\delta Q}{\delta u^\alpha}+\theta_\alpha^1\frac{\delta Q}{\delta\theta_\alpha}\right)\mathrm{d}x\\
&=\int\sum_{s\geq 0}\left(u^{\alpha,s+1}\frac{\partial\tilde{Q}}{\partial u^{\alpha,s}}+\theta_\alpha^{s+1}\frac{\partial\tilde{Q}}{\partial\theta_\alpha^s}\right)\mathrm{d}x\\
&=\int(\partial(Q))\,\mathrm{d}x=0,
\end{aligned}
$$

so ω is in the center of the graded Lie algebra $(\hat{\mathcal{F}},[\,,])$.

Suppose $P\in\hat{\mathcal{F}}^p$ is in the center of $(\hat{\mathcal{F}},[\,,])$, then for any $F\in\mathcal{F}=\hat{\mathcal{F}}^0$ we have $[P,F]=0$. Consider the action of $[P,F]$ on $F_2,\dots,F_p\in\mathcal{F}$,

$$[P,F](F_2,\dots,F_p)=\jmath(P)(F,F_2,\dots,F_p)=0,$$

so $P\in\mathrm{Ker}(\jmath)=\mathbb{R}\omega$. The lemma is proved. □

For more properties of the higher Euler operators and their generalizations, please refer to [24, 29] and the references therein.

Proof of Proposition 2.9: Suppose $P\in\Lambda^p$, $Q\in\Lambda^q$, take $P'\in\hat{\mathcal{F}}^p$, $Q'\in\hat{\mathcal{F}}^q$ such that

$$P=\jmath(P'),\quad Q=\jmath(Q'),$$

then define $[P,Q]'=\jmath([P',Q'])$. According to the above two lemmas, this definition is independent of the choice of P' and Q'. Lemma 2.11 shows that the operation $[\,,]'$ must coincides with the Nijenhuis-Richardson bracket $[\,,]$, so we have $[P,Q]\in\Lambda^{p+q-1}$. The proposition is proved. □

Lemma 2.10 shows that $\hat{\mathcal{F}}^p$ and Λ^p can be identified, except the $p = 1$ case. When $p = 1$, Lemma 2.7 shows that $\Lambda^1 \cong \mathcal{E}/\mathbb{R}\omega$, so we can identify $\mathcal{E}$ and $\hat{\mathcal{F}}^1$ as follow

$$X = (X^\alpha) \in \mathcal{E} \quad \leftrightarrow \quad X = \int (X^\alpha \theta_\alpha) \mathrm{d}x \in \hat{\mathcal{F}}^1.$$

It is easy to see that the action of $X \in \mathcal{E} = \hat{\mathcal{F}}^1$ on $F \in \mathcal{F} = \hat{\mathcal{F}}^0$ is exactly given by $[X, F]$. From now on, we will always working with $\hat{\mathcal{F}}$, and forget about $\mathcal{F}$, $\mathcal{E}$, $\mathcal{V}$, and Λ^*.

Definition 2.12 *An element $X \in \hat{\mathcal{F}}^1$ is called an evolutionary PDE. An element $F \in \hat{\mathcal{F}}^0$ is called a conserved quantity of X if $[X, F] = 0$. An element $P \in \hat{\mathcal{F}}^2$ is called a Hamiltonian structure if $[P, P] = 0$. An evolutionary PDE X is called Hamiltonian if there is a Hamiltonian structure P and a conserved quantity F such that $X = [P, F]$.*

3 Hamiltonian structures

3.1 Presentations and examples

Let $P = \int \tilde{P} \mathrm{d}x \in \hat{\mathcal{F}}^2$ be a variational bivector, then $\tilde{P}$ satisfies the following homogeneous condition

$$\tilde{P} = \frac{1}{2} \sum_{s \geq 0} \theta_\alpha^s \frac{\partial \tilde{P}}{\partial \theta_\alpha^s},$$

so we have

$$P = \frac{1}{2} \int \left(\sum_{s \geq 0} \theta_\alpha^s \frac{\partial \tilde{P}}{\partial \theta_\alpha^s} \right) \mathrm{d}x = \frac{1}{2} \int \left(\theta_\alpha \frac{\delta P}{\delta \theta_\alpha} \right) \mathrm{d}x.$$

Suppose

$$\frac{\delta P}{\delta \theta_\alpha} = \sum_{s \geq 0} P_s^{\alpha\beta} \theta_\beta^s = \left(\sum_{s \geq 0} P_s^{\alpha\beta} \partial^s \right) \theta_\beta, \tag{3.1}$$

then

$$P = \frac{1}{2} \int \left(\theta_\alpha \left(\sum_{s \geq 0} P_s^{\alpha\beta} \partial^s \right) \theta_\beta \right) \mathrm{d}x, \tag{3.2}$$

so a variational bivector corresponds to a matrix differential operator

$$\mathcal{P} = \left(\mathcal{P}^{\alpha\beta} \right) = \left(\sum_{s \geq 0} P_s^{\alpha\beta} \partial^s \right). \tag{3.3}$$

By computing the variational derivative of both side of (3.2) with respect to θ_α, one can show that

$$\mathcal{P} + \mathcal{P}^\dagger = 0, \tag{3.4}$$

where

$$\mathcal{P}^\dagger = \left((\mathcal{P}^\dagger)^{\alpha\beta} \right) = \left(\sum_{s \geq 0} (-\partial)^s P_s^{\beta\alpha} \right).$$

It is easy to see that the variational bivectors are one-to-one corresponding to the matrix differential operators (3.3) satisfying the condition (3.4), so we have the following definition.

Definition 3.1 *Let $P \in \hat{\mathcal{F}}^2$ be a Hamiltonian structure, the matrix differential operator $\mathcal{P}$ defined by* (3.1) *is called the Hamiltonian operator of P.*

In literatures, a Hamiltonian structure is often given by its Hamiltonian operator.

Let $P \in \hat{\mathcal{F}}^2$ be a Hamiltonian structure, then the bracket operation

$$\{\,,\}_P : \mathcal{F} \times \mathcal{F} \to \mathcal{F}, (F, G) \mapsto \{F, G\}_P = P(F, G)$$

is a Lie bracket, whose action can be computed explicitly:

$$\{F, G\}_P = P(F, G) = [[P, F], G] = \int \left(\frac{\delta f}{\delta u^\alpha} \mathcal{P}^{\alpha\beta} \left(\frac{\delta G}{\delta u^\beta} \right) \right) \mathrm{d}x.$$

If we enlarge the space $\mathcal{F}$ to contain functionals of the following form

$$u^\alpha(y) = \int_{S^1} u^\alpha(x)\delta(x - y)\mathrm{d}x,$$

then we have

$$\begin{aligned} \{u^\alpha(y), u^\beta(z)\}_P &= \int_{S^1} \delta(x - y)\mathcal{P}^{\alpha\beta}(u(x))\delta(x - z)\mathrm{d}x \\ &= \sum_{s \geq 0} P_s^{\alpha\beta}(u(y))\delta^{(s)}(y - z). \end{aligned}$$

This is another common way to present a Hamiltonian structure. We can call it the coordinate presentation.

Example 3.2 *Suppose $M = \mathbb{R}$, so $n = 1$. We can omit the α index.*

Let $P = \frac{1}{2} \int g(u)\theta\theta^1 \mathrm{d}x \in \hat{\mathcal{F}}^2$, then we have

$$\begin{aligned} \frac{\delta P}{\delta u} &= \frac{1}{2} g'(u)\theta\theta^1, \\ \frac{\delta P}{\delta \theta} &= \frac{1}{2} \left(g(u)\theta^1 + \partial\left(g(u)\theta\right) \right) = g(u)\theta^1 + \frac{1}{2} g'(u)u^1\theta, \end{aligned}$$

so we have

$$[P, P] = 2 \int \frac{\delta P}{\delta \theta} \frac{\delta P}{\delta u} \mathrm{d}x = 0,$$

so P is a Hamiltonian structure. The Hamiltonian operator reads

$$\mathcal{P} = g(u)\partial + \frac{1}{2} g'(u)u^1,$$

and the coordinate presentation reads

$$\{u(y), u(z)\} = g(u(y))\delta'(y - z) + \frac{1}{2} g'(u(y))u'_y\delta(y - z).$$

Consider a deformation of P:

$$\tilde{P} = P + c\int \theta\theta^3 \mathrm{d}x,$$

then

$$[P,P] = 2\int \frac{\delta P}{\delta\theta}\frac{\delta P}{\delta u}\mathrm{d}x = 2c\int g'(u)\theta\theta^1\theta^3\mathrm{d}x.$$

It is easy to see that $[P,P]=0$ if and only if $g''(u)=0$. So we obtain a family of Hamiltonian operators with three parameters a,b,c:

$$\mathcal{P}_{a,b,c} = (a\,u+b)\partial + \frac{a}{2}u^1 + c\,\partial^3.$$

In particular, the operators

$$\mathcal{P}_1 = \mathcal{P}_{0,1,0} = \partial,$$
$$\mathcal{P}_2 = \mathcal{P}_{1,0,\hbar/8} = u\partial + \frac{1}{2}u^1 + \frac{\hbar}{8}\partial^3$$

give the two Hamiltonian structures of the Korteweg-de Vries equation:

$$u_t = u\,u_x + \frac{\hbar}{12}u_{xxx}.$$

And the operators

$$\mathcal{P}_1 = \mathcal{P}_{0,1,-1} = \partial - \partial^3,$$
$$\mathcal{P}_2 = \mathcal{P}_{1,0,0} = u\partial + \frac{1}{2}u^1$$

give the two Hamiltonian structures of the Camassa-Holm equation:

$$u_t - u_{xxt} = 3\,u\,u_x - 2\,u_x\,u_{xx} - u\,u_{xxx}.$$

Example 3.3 *Let $M=\mathbb{R}^2$, we denote*

$$u^1 = u, \quad u^2 = v, \quad \theta_1 = \theta, \quad \theta_2 = \phi.$$

Define a series of shift operators

$$\mathcal{S}^k = e^{k\varepsilon\partial}, \quad k\in\mathbb{Z},$$

and denote by $a^\pm = \mathcal{S}^{\pm 1}(a)$, $a^{[k]} = \mathcal{S}^k(a)$ for $a\in\hat{\mathcal{A}}$, $k\in\mathbb{Z}$. The Toda equation (2.2) and (2.3) can be written as

$$u_t = e^{v^+} - e^v, \quad v_t = u - u^-.$$

Here we take $\varepsilon = 1$ for convenience.

The second Hamiltonian structure of the Toda equation can be written as

$$P_2 = \int \left(e^{v^+}\theta\theta^+ + u\theta(\phi^+ - \phi) + \phi\phi^+\right)\mathrm{d}x.$$

Its variational derivatives read

$$\begin{aligned}
\frac{\delta P_2}{\delta u} &= \theta(\phi^+ - \phi),\\
\frac{\delta P_2}{\delta v} &= e^v\theta^-\theta,\\
\frac{\delta P_2}{\delta \theta} &= e^{v^+}\theta^+ - e^v\theta^- + u(\phi^+ - \phi),\\
\frac{\delta P_2}{\delta \phi} &= u\theta - u^-\theta^- + \phi^+ - \phi^-.
\end{aligned}$$

Here we used the identity:

$$\frac{\delta F}{\delta z} = \sum_{k\in\mathbb{Z}} \mathcal{S}^{-k}\frac{\partial f}{\partial z^{[k]}},$$

where $F = \int f \mathrm{d}z \in \hat{\mathcal{F}}$, $z = u, v, \theta, \phi$.

Then, by using the following fact

$$\int a\,\mathrm{d}x = \int a^{[k]}\,\mathrm{d}x, \quad \textit{for all } a \in \hat{\mathcal{A}},\ k \in \mathbb{Z},$$

we obtain

$$\begin{aligned}
&\frac{1}{2}[P_2, P_2]\\
=&\int \Big(\Big(e^{v^+}\theta^+ - e^v\theta^- + u(\phi^+ - \phi)\Big)\,\theta(\phi^+ - \phi)\\
&\big(u\theta - u^-\theta^- + \phi^+ - \phi^-\big)\,e^v\theta^-\theta\Big)\,\mathrm{d}x\\
=&\int \Big(e^{v^+}\theta^+\theta\phi^+ - e^{v^+}\theta^+\theta\phi + e^v\theta^-\theta\phi - e^v\theta^-\theta\phi^-\Big)\,\mathrm{d}x\\
=&\int \big(e^v\theta\theta^-\phi - e^v\theta\theta^-\phi^- + e^v\theta^-\theta\phi - e^v\theta^-\theta\phi^-\big)\,\mathrm{d}x\\
=&0,
\end{aligned}$$

so P_2 *is indeed a Hamiltonian structure.*

The first Hamiltonian structure of the Toda equation can be written as

$$P_1 = \int \big(\theta(\phi^+ - \phi)\big)\,\mathrm{d}x.$$

One can show its hamiltonianily by using a similar method.

The two Hamiltonian operators read

$$\begin{aligned}
\mathcal{P}_1 &= \begin{pmatrix} 0 & \mathcal{S} - 1 \\ 1 - \mathcal{S}^{-1} & 0 \end{pmatrix},\\
\mathcal{P}_2 &= \begin{pmatrix} \mathcal{S}e^v - e^v\mathcal{S}^{-1} & u(\mathcal{S} - 1) \\ (1 - \mathcal{S}^{-1})u & \mathcal{S} - \mathcal{S}^{-1} \end{pmatrix}.
\end{aligned}$$

The coordinate presentations can be also written down by acting the above operators on δ*-functions.*

3.2 Miura transformations

Consider the follow equations:

$$\text{KdV}: \quad u_t - 6uu_x + u_{xxx} = 0,$$
$$\text{mKdV}: \quad v_t - 6v^2 v_x + v_{xxx} = 0.$$

Miura found that if v is a solution to the mKdV equation, then $u = v^2 + v_x$ gives a solution of the KdV equation.

In general, for an evolutionary PDE

$$u_t^\alpha = X^\alpha, \quad \text{where } X^\alpha \in \mathcal{A},$$

we can transform it to another equation by using transformations of the following form:

$$u^\alpha \mapsto \tilde{u}^\alpha = F^\alpha(u) + Y^\alpha$$

where $F^\alpha(u)$ is a local diffeomorphism, and $Y^\alpha \in \mathcal{A}_{>0}$. We also call them Miura transformations.

Miura transformations are important for Gromov-Witten theory. For example, when considering the target space $\mathbb{P}^1$, the corresponding integrable system is the extended Toda hierarchy, whose equations (like (2.2) and (2.3)) contain $\varepsilon = \sqrt{\hbar}$. On the other hand, the free energy and two-point functions of this model should be formal Laurent series of $\hbar$, so we need to perform certain Miura transformations to eliminate the terms containing odd powers of ε.

It is easy to see that any Miura transformation can be written as the composition of a local diffeomorphism and a Miura transformation of the following form:

$$u^\alpha \mapsto \tilde{u}^\alpha = u^\alpha + Y^\alpha.$$

Local diffeomorphisms are just coordinates transformation on the manifold M, which is easy to deal with. For Miura transformations of the above form, we have the following lemmas.

Lemma 3.4 *For any $Y^\alpha \in \mathcal{A}_{>0}$ ($\alpha = 1, \dots, n$), there exists a variational vector $Z \in \hat{\mathcal{F}}^1_{>0}$ such that*

$$\tilde{u}^\alpha = u^\alpha + Y^\alpha = e^{D_Z}(u^\alpha),$$

where D_Z is the derivation defined by (2.7). Z is called the generator of this Miura transformation.

Proof: Let $\nu = \min\{\nu(Y^\alpha) \mid \alpha = 1, \dots, n\} > 0$. Write Y^α as sum of its homogeneous components

$$Y^\alpha = Y^\alpha_\nu + Y^\alpha_{\nu+1} + \cdots .$$

Take $Z_{(1)} = \int (Y^\alpha_\nu \theta_\alpha)\, \mathrm{d}x \in \hat{\mathcal{F}}^1_\nu$, then we have

$$\begin{aligned} & e^{-D_{Z_{(1)}}}(u^\alpha + Y^\alpha) \\ =& u^\alpha + Y^\alpha_\nu + Y^\alpha_{\nu+1} + \cdots \\ & - \left(Y^\alpha_\nu + D_{Z_{(1)}}(Y^\alpha_{\nu+1}) + \cdots\right) + \cdots \\ =& u^\alpha + \tilde{Y}^\alpha, \end{aligned}$$

where $\nu(\tilde{Y}^\alpha) \geq \nu + 1$.

For $\tilde{Y}^\alpha$, we can take a $Z_{(2)} \in \hat{\mathcal{F}}^1_{\nu+1}$, such that

$$e^{-D_{Z_{(2)}}}(u^\alpha + \tilde{Y}^\alpha) = u^\alpha + \tilde{\tilde{Y}}^\alpha,$$

where $\nu(\tilde{\tilde{Y}}^\alpha) \geq \nu + 2$.

So we obtain a series of variational vector $Z_{(1)}, Z_{(2)}, \dots \in \hat{\mathcal{F}}^1$ such that

i) $\nu(Z_{(1)}) < \nu(Z_{(2)}) < \cdots$,

ii) $u^\alpha + Y^\alpha = e^{D_{Z_{(1)}}} e^{D_{Z_{(2)}}} \cdots (u^\alpha)$.

Then, by using the Baker-Campbell-Hausdorff formula and the commuting relation (2.8), one can show that there exist $Z \in \hat{\mathcal{F}}^1_{>0}$ such that $u^\alpha + Y^\alpha = e^{D_Z}(u^\alpha)$. The lemma is proved. □

Lemma 3.5 *Let $u^\alpha \mapsto \tilde{u}^\alpha = u^\alpha + Y^\alpha$ be a Miura transformation with generator $Z \in \hat{\mathcal{F}}^1$, then this Miura transformation transforms $P \in \hat{\mathcal{F}}^p$ to $e^{-\mathrm{ad}_Z}(P)$. We name Miura transformations of this form gauge transformations.*

This lemma depends on a transformation formula of variational derivatives, whose proof cannot be given here, so we omit it. One can find a full proof in §2.5 of [29].

In Poisson geometry, Darboux theorem plays an important role, which classifies the equivalence classes of Poisson structures modulo local coordinates transformations. We have a similar problem for the infinite dimensional case. Let $\mathcal{H}$ be the set of Hamiltonian structures

$$\mathcal{H} = \{P \in \hat{\mathcal{F}}^2 \mid [P, P] = 0\},$$

and $\mathcal{G}$ be the group of gauge transformations

$$\mathcal{G} = \{e^{\mathrm{ad}_Z} \mid Z \in \hat{\mathcal{F}}^1_{>0}\},$$

then $\mathcal{G}$ acts on $\mathcal{H}$, and the corresponding Darboux theorem is a certain description of the quotient space $\mathcal{H}/\mathcal{G}$.

A classification problem is often converted to a deformation problem. For a Hamiltonian structure $P \in \mathcal{H}$, let $\nu = \nu(P)$, and write P as

$$P = P_0 + Q, \quad \text{where } P_0 \in \hat{\mathcal{F}}^2_\nu,\ \nu(Q) > \nu,$$

then P_0 must be a Hamiltonian structure. We call it the leading term of P. Then the equation $[P, P] = 0$ can be written as

$$d_{P_0}(Q) + \frac{1}{2}[Q, Q] = 0, \tag{3.5}$$

where $d_{P_0} = \mathrm{ad}_{P_0}$. The equation (3.5) is called the Maurer-Cartan equation for P_0, and a solution to it is called a Maurer-Cartan element for P_0.

Let $\mathcal{MC}(P)$ be the set of Maurer-Cartan elements for a homogeneous Hamiltonian structure $P \in \hat{\mathcal{F}}^2_\nu$:

$$\mathcal{MC}(P) = \{Q \in \hat{\mathcal{F}}^2_{>\nu} \mid d_P(Q) + \frac{1}{2}[Q, Q] = 0\},$$

then $\mathcal{G}$ also acts on $\mathcal{MC}(P)$:

$$(e^{\mathrm{ad}_Z}, Q) \mapsto \tilde{Q} = e^{\mathrm{ad}_Z}(P+Q) - P.$$

The deformation problem is just to ask the structure of the quotient space $\mathcal{MC}(P)/\mathcal{G}$.

The following definition and lemma are very standard in deformation theory, so we omit their proof.

Definition 3.6 *Let $P \in \hat{\mathcal{F}}^2_\nu$ be a homogeneous Hamiltonian structure.*

(a) $Q \in \hat{\mathcal{F}}^2_{>\nu}$ is called a infinitesimal deformation of P if $d_P(Q) = 0$.

(b) Two infinitesimal deformation Q_1, Q_2 are called equivalent if there exists $Z \in \hat{\mathcal{F}}^1_{>0}$ such that $Q_1 - Q_2 = d_P(Z)$.

(c) An infinitesimal deformation Q is called trivial if it is equivalent to 0.

(d) The triple $(\hat{\mathcal{F}}, [\,,], d_P)$ forms a differential graded Lie algebra (DGLA). Its cohomology is defined as

$$H(\hat{\mathcal{F}}, d_P) = \mathrm{Ker}(d_P)/\mathrm{Im}(d_P).$$

Note that P is homogeneous, so we have the following decomposition

$$H(\hat{\mathcal{F}}, d_P) = \bigoplus_{p\geq 0}\bigoplus_{d\geq 0} H^p_d(\hat{\mathcal{F}}, P),$$

where

$$H^p_d(\hat{\mathcal{F}}, d_P) = \frac{\mathrm{Ker}(d_P : \hat{\mathcal{F}}^p_d \to \hat{\mathcal{F}}^{p+1}_{d+\nu})}{\mathrm{Im}(d_P : \hat{\mathcal{F}}^{p-1}_{d-\nu} \to \hat{\mathcal{F}}^p_d)}.$$

Lemma 3.7 *Let $P \in \hat{\mathcal{F}}^2_\nu$ be a homogeneous Hamiltonian structure.*

(a) The space of equivalence classes of infinitesimal deformations of P is given by $H^2_{>\nu}(\hat{\mathcal{F}}, d_P)$. In particular, every deformation of P is trivial if and only if $H^2_{>\nu}(\hat{\mathcal{F}}, d_P)$ vanishes.

(b) Let ν' be the lowest degree of classes in $H^2_{>\nu}(\hat{\mathcal{F}}, d_P)$. If $H^3_{\geq 2\nu'}(\hat{\mathcal{F}}, d_P)$ vanishes, then every infinitesimal deformation can be extended to a genuine deformation, and the space of equivalence classes of deformations of P is just $H^2_{>\nu}(\hat{\mathcal{F}}, d_P)$.

Example 3.8 *Let $P = \frac{1}{2}\int \left(P^{\alpha\beta}(u)\theta_\alpha\theta_\beta\right) \mathrm{d}x \in \hat{\mathcal{F}}^2_0$ be a Hamiltonian structure. Then it is easy to see that $(P^{\alpha\beta}(u)\frac{\partial}{\partial u^\alpha} \wedge \frac{\partial}{\partial u^\alpha})$ gives a Poisson structure on the manifold M. We assume that $\det(P^{\alpha\beta}) \neq 0$, then, according to the Darboux theorem in finite dimensional symplectic geometry, there exists a local coordinate system $(u^1, \ldots, u^n)$ such that $(P^{\alpha\beta})$ is a constant matrix. Let $\theta^\alpha = P^{\alpha\beta}\theta_\beta$, then*

$$\frac{\delta P}{\delta u^\alpha} = 0, \quad \frac{\delta P}{\delta \theta_\alpha} = \theta^\alpha.$$

The operator D_P (see (2.7)) reads

$$D_P = \sum_{s\geq 0} \partial^s\theta^\alpha \frac{\partial}{\partial u^{\alpha,s}}.$$

If we write $\partial^s\theta^\alpha$ as $\mathrm{d}u^{\alpha,s}$, then D_P is just the de Rham differential of $J^\infty(M)$. In particular, $D_P^2 = 0$, so we have a complex $(\hat{\mathcal{A}}, D_P)$.

By definition, the following sequence of complex morphisms is exact

$$0 \to (\hat{\mathcal{A}}/\mathbb{R}, D_P) \xrightarrow{\partial} (\hat{\mathcal{A}}, D_P) \xrightarrow{\int} (\hat{\mathcal{F}}, d_P) \to 0,$$

so we have a long exact sequence of cohomologies

$$\begin{aligned} \cdots \to H^p_{d-1}(\hat{\mathcal{A}}/\mathbb{R}, D_P) \to H^p_d(\hat{\mathcal{A}}, D_P) \to H^p_d(\hat{\mathcal{F}}, d_P) \\ \to H^{p+1}_{d-1}(\hat{\mathcal{A}}/\mathbb{R}, D_P) \to H^{p+1}_d(\hat{\mathcal{A}}, D_P) \to H^{p+1}_d(\hat{\mathcal{F}}, d_P) \to \cdots . \end{aligned}$$

Define a map

$$F : [0,1] \times J^\infty(\hat{M}), \quad (t, (u^{\alpha,s}, \theta^s_\alpha)) \mapsto (t^s\, u^{\alpha,s}, t^s\, \theta^s_\alpha),$$

which induces a homotopy equivalence from the complex $(\hat{\mathcal{A}}, D_P)$ to the de Rham complex $(\Omega^*(M), \mathrm{d}_{dR})$ of M, so we have

$$H^p_d(\hat{\mathcal{A}}, D_P) \cong \begin{cases} H^p_{dR}(M), & d = 0; \\ 0, & d > 0. \end{cases}$$

Similarly,

$$H^p_d(\hat{\mathcal{A}}/\mathbb{R}, D_P) \cong \begin{cases} H^0_{dR}(M)/\mathbb{R}, & p = 0,\ d = 0; \\ H^p_{dR}(M), & p > 0,\ d = 0; \\ 0, & d > 0. \end{cases}$$

So we have

$$H^p_d(\hat{\mathcal{F}}, d_P) \cong \begin{cases} H^p_{dR}(M), & d = 0; \\ H^{p+1}_{dR}(M), & d = 1; \\ 0, & d \geq 2. \end{cases}$$

In particular, if $H^3_{dR}(M) \cong 0$, every deformation of P is trivial.

If $H^3_{dR}(M) \ncong 0$, there are non-trivial infinitesimal deformations, which can be always extended to a genuine deformation since $H^3_{\geq 2}(\hat{\mathcal{F}}, d_P) \cong 0$. For example, if G is a simple compact Lie group, and $M = T^*G$, then M has the canonical symplectic structure, and $H^3_{dR}(M) \cong H^3_{dR}(G) \ncong 0$, so there is a non-trivial infinitesimal deformation with degree one. The Drinfeld-Sokolov Hamiltonian structure can be regarded as a reduction of this deformation.

3.3 Hydrodynamic Hamiltonian structures

In this subsection, we consider homogeneous Hamiltonian structures with degree one.

Lemma 3.9 ([9]) *Let $P \in \hat{\mathcal{F}}^2_1$ be a variational bivector, the corresponding matrix differential operator reads*

$$\mathcal{P}^{\alpha\beta} = g^{\alpha\beta}(u)\partial + \Gamma^{\alpha\beta}_\gamma(u) u^{\gamma,1}.$$

Suppose $\det(g^{\alpha\beta}) \neq 0$, then P is a Hamiltonian structure if and only if the following two conditions hold true:

i) $g = (g_{\alpha\beta}) = (g^{\alpha\beta})^{-1}$ is a flat (not necessary positive definite) metric on M.

ii) $\Gamma^\gamma_{\alpha\beta} = -g_{\alpha\sigma}\Gamma^{\sigma\gamma}_\beta$ *give the Christoffel symbols of the Levi-Civita connection of* g*.*

Proof: The bivector $P \in \hat{\mathcal{F}}^2_1$ reads

$$P = \frac{1}{2}\int \left(g^{\alpha\beta}\theta_\alpha\theta^1_\beta + \Gamma^{\alpha\beta}_\gamma u^{\gamma,1}\theta_\alpha\theta_\beta\right)\mathrm{d}x.$$

The skew-symmetry condition $\mathcal{P} + \mathcal{P}^\dagger = 0$ gives

$$g^{\alpha\beta} = g^{\beta\alpha}, \tag{3.6}$$

$$\Gamma^{\alpha\beta}_\gamma + \Gamma^{\beta\alpha}_\gamma = \frac{\partial g^{\alpha\beta}}{\partial u^\gamma}. \tag{3.7}$$

The variational derivatives of P read

$$\frac{\delta P}{\delta u^\sigma} = \Gamma^{\beta\alpha}_\sigma\theta_\alpha\theta^1_\beta + \frac{1}{2}\left(\frac{\partial\Gamma^{\alpha\beta}_\gamma}{\partial u^\sigma} - \frac{\partial\Gamma^{\alpha\beta}_\sigma}{\partial u^\gamma}\right)u^{\gamma,1}\theta_\alpha\theta_\beta,$$
$$\frac{\delta P}{\delta\theta_\sigma} = g^{\sigma\beta}\theta^1_\beta + \Gamma^{\sigma\beta}_\gamma u^{\gamma,1}\theta_\beta.$$

Let $W = \frac{1}{2}[P, P]$, then we have

$$W = \int \left(A^{\alpha\beta\gamma}\theta_\alpha\theta^1_\beta\theta^1_\gamma + B^{\alpha\beta\gamma}_\sigma u^{\sigma,1}\theta_\alpha\theta_\beta\theta^1_\gamma + C^{\alpha\beta\gamma}_{\sigma_1\sigma_2}u^{\sigma_1,1}u^{\sigma_2,1}\theta_\alpha\theta_\beta\theta_\gamma\right)\mathrm{d}x$$

where

$$A^{\alpha\beta\gamma} = g^{\gamma\sigma}\Gamma^{\alpha\beta}_\sigma,$$
$$B^{\alpha\beta\gamma}_\sigma = \frac{1}{2}g^{\gamma\delta}\left(\frac{\partial\Gamma^{\alpha\beta}_\sigma}{\partial u^\delta} - \frac{\partial\Gamma^{\alpha\beta}_\delta}{\partial u^\sigma}\right) + \Gamma^{\delta\alpha}_\sigma\Gamma^{\gamma\beta}_\delta,$$
$$C^{\alpha\beta\gamma}_{\sigma_1\sigma_2} = \frac{1}{2}\Gamma^{\gamma\delta}_{\sigma_2}\left(\frac{\partial\Gamma^{\alpha\beta}_{\sigma_1}}{\partial u^\delta} - \frac{\partial\Gamma^{\alpha\beta}_\delta}{\partial u^{\sigma_1}}\right).$$

If $W = 0$ then $\frac{\delta W}{\delta\theta_\alpha} = 0$ for all $\alpha = 1, \dots, n$, so we have

$$0 = \frac{\partial}{\partial\theta^2_\beta}\frac{\delta W}{\delta\theta_\alpha} = \frac{\partial}{\partial\theta^2_\beta}\left(\frac{\partial\tilde{W}}{\partial\theta_\alpha} - \partial\frac{\partial\tilde{W}}{\partial\theta^1_\alpha}\right) = \frac{\partial^2\tilde{W}}{\partial\theta^1_\alpha\partial\theta^1_\beta}$$
$$0 = \frac{\partial}{\partial u^{\beta,2}}\frac{\delta W}{\delta\theta_\alpha} = \frac{\partial}{\partial u^{\beta,2}}\left(\frac{\partial\tilde{W}}{\partial\theta_\alpha} - \partial\frac{\partial\tilde{W}}{\partial\theta^1_\alpha}\right) = -\frac{\partial^2\tilde{W}}{\partial\theta^1_\alpha\partial u^{\beta,1}},$$

where $\tilde{W}$ is the density given above. The above two identities imply that

$$A^{\alpha\beta\gamma} = A^{\alpha\gamma\beta}, \tag{3.8}$$

$$B^{\alpha\beta\gamma}_\sigma = B^{\beta\alpha\gamma}_\sigma. \tag{3.9}$$

The equation (3.6) shows that g can be regarded as a metric. The equation (3.7) shows that the metric g is invariant with respect to the connection defined by $\Gamma^\gamma_{\alpha\beta}$. The equation (3.8) shows that this connection is torsion-free, so it must be the Levi-Civita connection of g. The last equation (3.9) is equivalent to the flatness of this connection.

Conversely, if g and Γ satisfy the condition i) and ii), we can choose a system of flat coordinates such that g is a constant metric and Γ vanish, then it is easy to show that P is a Hamiltonian structure. The lemma is proved. □

Definition 3.10 *A Hamiltonian structure $P \in \hat{\mathcal{F}}_1^2$ is called of hydrodynamic type if it satisfies the conditions in Lemma 3.9.*

According to Lemma 3.9, we can always choose a coordinate system such that

$$P = \frac{1}{2}\int \left(\eta^{\alpha\beta}\theta_\alpha\theta_\beta^1\right) \mathrm{d}x, \tag{3.10}$$

where $(\eta^{\alpha\beta})$ is a constant symmetric non-degenerate matrix.

From now on, we assume that M is connected and contractible, then consider the deformation problem of (3.10). The computation is similar to the degree zero case. The variational derivatives read

$$\frac{\delta P}{\delta u^\alpha} = 0, \quad \frac{\delta P}{\delta \theta_\alpha} = \eta^{\alpha\beta}\theta_\beta^1.$$

We denote $\theta^{\alpha,s} = \eta^{\alpha\beta}\theta_\beta^s$, then the operator D_P reads

$$D_P = \sum_{s\geq 0} \theta^{\alpha,s+1}\frac{\partial}{\partial u^{\alpha,s}}.$$

The algebra $\hat{\mathcal{A}}$ can be decomposed as $\hat{\mathcal{A}} = \hat{\mathcal{A}}' \otimes \hat{\mathcal{A}}''$, where

$$\begin{aligned}
\hat{\mathcal{A}}' =& \mathcal{A} \otimes \wedge^* \left(\mathrm{Span}_{\mathbb{R}}\left\{\theta^{\alpha,s} \mid \alpha = 1,\ldots,n;\ s \geq 1\right\}\right),\\
\hat{\mathcal{A}}'' =& \wedge^* \left(\mathrm{Span}_{\mathbb{R}}\left\{\theta^{1,0},\ldots,\theta^{n,0}\right\}\right).
\end{aligned}$$

Note that $D_P(\hat{\mathcal{A}}'') = 0$, so we have $(\hat{\mathcal{A}}, D_P) = (\hat{\mathcal{A}}', D_P) \otimes \hat{\mathcal{A}}''$, and

$$H^*(\hat{\mathcal{A}}, D_P) = H^*(\hat{\mathcal{A}}', D_P) \otimes \hat{\mathcal{A}}''.$$

On the other hand, if we replace $\theta^{\alpha,s+1}$ by $\mathrm{d}u^{\alpha,s}$, then $(\hat{\mathcal{A}}', D_P)$ is again the de Rham complex of $J^\infty(M)$, so we have

$$H_d^p(\hat{\mathcal{A}}', D_P) = \begin{cases} \mathbb{R}, & (p,d) = (0,0);\\ 0, & (p,d) \neq (0,0),\end{cases}$$

which imply

$$H_d^p(\hat{\mathcal{A}}, D_P) = \begin{cases} \wedge^p(\mathbb{R}^n), & d = 0;\\ 0, & d > 0.\end{cases}$$

Then it is easy to see

$$H_d^p(\hat{\mathcal{A}}/\mathbb{R}, D_P) = \begin{cases} \wedge^p(\mathbb{R}^n), & p > 0, d = 0;\\ 0, & \text{otherwise.}\end{cases}$$

Finally, by using the long exact sequence

$$\begin{aligned}
\cdots &\to H_{d-1}^p(\hat{\mathcal{A}}/\mathbb{R}, D_P) \to H_d^p(\hat{\mathcal{A}}, D_P) \to H_d^p(\hat{\mathcal{F}}, d_P)\\
&\to H_d^{p+1}(\hat{\mathcal{A}}/\mathbb{R}, D_P) \to H_{d+1}^{p+1}(\hat{\mathcal{A}}, D_P) \to H_{d+1}^{p+1}(\hat{\mathcal{F}}, d_P) \to \cdots,
\end{aligned}$$

we obtain

$$H^p_d(\hat{\mathcal{F}}, d_P) = \begin{cases} \wedge^p(\mathbb{R}^n) \oplus \wedge^{p+1}(\mathbb{R}^n), & d = 0; \\ 0, & d > 0. \end{cases}$$

In particular, $H^2_{>0}(\hat{\mathcal{F}}, d_P) \cong 0$, so there is no non-trivial deformation of P. This gives the Darboux theorem for Hamiltonian structures of hydrodynamic type.

Theorem 3.11 *Let $P \in \hat{\mathcal{F}}^2_1$ be a Hamiltonian structure of hydrodynamic type, then for any deformation $\tilde{P} = P + Q$, there exists a gauge transformation e^{ad_Z} such that $e^{\mathrm{ad}_Z}(\tilde{P}) = P$.*

It is interesting to ask whether there are Darboux theorems for Hamiltonian structures with degrees ≥ 2. For example, a degree two Hamiltonian operator has the following general form

$$\mathcal{P}^{\alpha\beta} = g^{\alpha\beta}\partial^2 + \Gamma^{\alpha\beta}_\gamma u^{\gamma,1}\partial + \left(P^{\alpha\beta}_\gamma u^{\gamma,2} + Q^{\alpha\beta}_{\xi\zeta} u^{\xi,1} u^{\zeta,1}\right).$$

We can assume that $g = \left(g^{\alpha\beta}\right)$ is non-degenerate, then g^{-1} is a symplectic structure on M. One can show that $\Gamma^{\alpha\beta}_\gamma$ is given by a symplectic connection of g^{-1}, and it should satisfy a certain flatness condition. But we know nothing about P and Q.

In [5], De Sole and Kac computed certain cohomology groups similar to $H^*(\hat{\mathcal{F}}, d_P)$ for $\mathcal{P}^{\alpha\beta} = g^{\alpha\beta}\partial^N$ with $g^{\alpha\beta}$ being constant and $\det(g^{\alpha\beta}) \neq 0$. Their definition is slightly different from ours, but the result is quite comparable (see [6] for details).

4 Bihamiltonian structures

4.1 Definition and semisimplicity

A bihamiltonian structure (P_1, P_2) is a pair of Hamiltonian structures such that $[P_1, P_2] = 0$.

Lemma 4.1 *Let $P \in \hat{\mathcal{F}}^2$ be a Hamiltonian structure, if there is an $X \in \hat{\mathcal{F}}^1$ such that $[X, [X, P]] = 0$, then $(P, [P, X])$ is a bihamiltonian structure. Bihamiltonian structures obtained by this way are called exact bihamiltonian structures.*

Proof: Let $Q = [P, X]$, then $[P, P] = 0$, $[P, Q] = 0$, and

$$[Q, Q] = [[P, X], Q] = -[[X, Q], P] - [[Q, P], X] = 0.$$

The lemma is proved. □

Example 4.2 *The KdV equation has two Hamiltonian structures*

$$P_1 = \int \theta\theta^1 \mathrm{d}x, \quad P_2 = \int \left(u\theta\theta^1 + \frac{\hbar}{8}\theta\theta^3\right) \mathrm{d}x.$$

Let $X = \int \theta \mathrm{d}x$, then $P_1 = [P_2, X]$, and $[P_1, X] = 0$, so (P_1, P_2) is indeed a bihamiltonian structure, and it is exact.

Example 4.3 *The Toda equation has two Hamiltonian structures*

$$P_1 = \int \theta(\phi^+ - \phi)\mathrm{d}x,$$
$$P_2 = \int \left(e^{v^+}\theta\theta^+ + u\theta(\phi^+ - \phi) + \phi\phi^+\right)\mathrm{d}x.$$

Let $X = \int \theta \mathrm{d}x$, then $P_1 = [P_2, X]$, and $[P_1, X] = 0$, so (P_1, P_2) is also a bihamiltonian structure, and it is exact.

Example 4.4 *The Camassa-Holm equation has two Hamiltonian structures:*

$$P_1 = \int \theta(\theta^1 - \theta^3)\mathrm{d}x, \quad P_2 = \int \left(u\theta\theta^1\right)\mathrm{d}x.$$

We have shown in the last section that any linear combination of P_1 and P_2 is a Hamiltonian structure, which implies that $[P_1, P_2] = 0$, so (P_1, P_2) is a bihamiltonian structure. Note that this bihamiltonian structure is not exact.

Let (P_1, P_2) be a bihamiltonian structure, if both P_1 and P_2 are of hydrodynamic type, then (P_1, P_2) is also called of hydrodynamic type. According to Lemma 3.9, there exists a pair of flat metric g_1 and g_2, such that

$$P_a = \frac{1}{2}\int \left(g_a^{\alpha\beta}(u)\theta_\alpha\theta_\beta^1 + \Gamma_{\gamma,a}^{\alpha\beta}u^{\gamma,1}\theta_\alpha\theta_\beta\right)\mathrm{d}x,$$

where $a = 1, 2$, and $g_a^{\alpha\beta}$ and $\Gamma_{\gamma,a}^{\alpha\beta}$ are given by the contravariant metric and the connection coefficients of g_a. In general, one cannot find a coordinate system such that both g_1 and g_2 are constant.

Definition 4.5 *Let (P_1, P_2) be a bihamiltonian structure of hydrodynamic type, whose contravariant metric are $g_1^{\alpha\beta}(u)$ and $g_2^{\alpha\beta}(u)$. If the roots*

$$\lambda^1(u), \ldots, \lambda^n(u)$$

of the characteristic equation

$$\det\left(g_2^{\alpha\beta}(u) - \lambda g_1^{\alpha\beta}(u)\right) = 0$$

are not constant and distinct, the bihamiltonian structure (P_1, P_2) is called semisimple. The roots $\lambda^1, \ldots, \lambda^n$ are called the canonical coordinates of (P_1, P_2).

Theorem 4.6 ([20]) *Let (P_1, P_2) be a semisimple bihamiltonian structure. Its canonical coordinates can serve as local coordinates near any point on M. Furthermore, the two metric has the following form in the canonical coordinates*

$$g_1^{ij} = \delta^{ij} f^i(\lambda), \quad g_2^{ij} = \delta^{ij}\lambda^i f^i(\lambda).$$

Note that we don't sum over repeated Latin indexes i, j.

In canonical coordinates, the two Hamiltonian structures have the following forms:

$$P_1 = \frac{1}{2}\int\left(\sum_{i=1}^n f^i(\lambda)\theta_i\theta_i^1 + \sum_{i,j=1}^n A^{ij}\theta_i\theta_j\right)\mathrm{d}x,$$
$$P_2 = \frac{1}{2}\int\left(\sum_{i=1}^n g^i(\lambda)\theta_i\theta_i^1 + \sum_{i,j=1}^n B^{ij}\theta_i\theta_j\right)\mathrm{d}x,$$

where $g^i(\lambda) = \lambda^i f^i(\lambda)$, and

$$A^{ij} = \frac{1}{2}\left(\frac{f^i}{f_j}\frac{\partial f^j}{\partial\lambda^i}\lambda^{j,1} - \frac{f^j}{f_i}\frac{\partial f^i}{\partial\lambda^j}\lambda^{i,1}\right),$$
$$B^{ij} = \frac{1}{2}\left(\frac{g^i}{f_j}\frac{\partial f^j}{\partial\lambda^i}\lambda^{j,1} - \frac{g^j}{f_i}\frac{\partial f^i}{\partial\lambda^j}\lambda^{i,1}\right).$$

Note that $f^i \neq 0$, $g^i \neq 0$, and A^{ij}, B^{ij} are skew-symmetric.

Example 4.7 ([7]) *Let M be a Frobenius manifold, then we have a pair of compatible metric*

$$g_1^{\alpha\beta} = \eta^{\alpha\beta}, \quad g_2^{\alpha\beta} = E^\gamma c_\gamma^{\alpha\beta},$$

which define a bihamiltonian structure (P_1, P_2) of hydrodynamic type. If M is semisimple, then (P_1, P_2) is also semisimple, and the canonical coordinates of (P_1, P_2) coincide with the ones of M. Bihamiltonian structures of Frobenius manifolds are always exact, because $P_1 = [P_2, e]$, where e is the unit vector field.

4.2 Bihamiltonian cohomology

In this subsection, we consider the deformation problem of a semisimple bihamiltonian structure.

Let (P_1, P_2) be a semisimple bihamiltonian structure, denote by $d_a = \mathrm{ad}_{P_a}$ $(a = 1, 2)$, then they satisfy

$$d_1^2 = 0, \quad d_1 d_2 + d_2 d_1 = 0, \quad d_2^2 = 0,$$

so we have a double complexes $(\hat{\mathcal{F}}^2, d_1, d_2)$.

A deformation of (P_1, P_2) is a bihamiltonian structure of the following form

$$(\tilde{P}_1, \tilde{P}_2) = (P_1 + Q_1, P_2 + Q_2),$$

where $Q_a \in \hat{\mathcal{F}}^2_{>1}$ $(a = 1, 2)$. According to the results given in Section 3.3, there is a gauge transformation e^{ad_Z} such that $e^{\mathrm{ad}_Z}(\tilde{P}_1) = P_1$, so we can take $Q_1 = 0$, and rename Q_2 to Q. Then $(\tilde{P}_1, \tilde{P}_2) = (P_1, P_2 + Q)$ is a bihamiltonian structure if and only if

$$d_1(Q) = 0, \quad d_2(Q) + \frac{1}{2}[Q, Q] = 0.$$

A bivector Q satisfying the above conditions is called a Maurer-Cartan element for (P_1, P_2), and we denote the set of Maurer-Cartan elements by $\mathcal{MC}(P_1, P_2)$:

$$\mathcal{MC}(P_1, P_2) = \{Q \in \hat{\mathcal{F}}^2_{>1} \mid d_1(Q) = 0,\ d_2(Q) + \frac{1}{2}[Q, Q] = 0\}.$$

Two deformations are equivalent if there exists a gauge transformation that convert one to another. Note that our P_1 is fixed, so the gauge transformations should preserve P_1, we denote such gauge transformations as $\mathcal{G}(P_1)$:

$$\mathcal{G}(P_1) = \{e^{\operatorname{ad}_Z} \mid Z \in \hat{\mathcal{F}}^1_{>0},\ d_1(Z) = 0\}.$$

The deformation problem for the bihamiltonian structure (P_1, P_2) is just to ask the structure of the quotient space $\mathcal{MC}(P_1, P_2)/\mathcal{G}(P_1)$.

Definition 4.8 *(a) $Q \in \hat{\mathcal{F}}^2_{>1}$ is called a infinitesimal deformation of (P_1, P_2) if $d_1(Q) = 0$, $d_2(Q) = 0$.*

(b) Two infinitesimal deformations Q_1, Q_2 are called equivalent if there exists $Z \in \hat{\mathcal{F}}^1_{>0}$ such that $d_1(Z) = 0$, $d_2(Z) = Q_1 - Q_2$.

(c) An infinitesimal deformation Q is called trivial if it is equivalent to 0.

(d) The bihamiltonian cohomologies of (P_1, P_2) are defined as

$$BH^p_d(\hat{\mathcal{F}}, d_1, d_2) = \frac{\hat{\mathcal{F}}^p_d \cap \operatorname{Ker}(d_1) \cap \operatorname{Ker}(d_2)}{\hat{\mathcal{F}}^p_d \cap \operatorname{Im}(d_1 d_2)}.$$

The following lemma is quite standard, so we omit its proof.

Lemma 4.9 *Let (P_1, P_2) be a semisimple bihamiltonian structure.*

(a) The cohomology group $BH^2_{>1}(\hat{\mathcal{F}}, d_1, d_2)$ gives the space of equivalence classes of infinitesimal deformations of (P_1, P_2).

(b) Let ν' be the lowest degree of classes in $BH^2_{>1}(\hat{\mathcal{F}}, d_1, d_2)$. If

$$BH^3_{\geq 2\nu'}(\hat{\mathcal{F}}, d_1, d_2) \cong 0,$$

then every infinitesimal deformation of (P_1, P_2) can be extended to a genuine deformation, and $BH^2_{>1}(\hat{\mathcal{F}}, d_1, d_2)$ actually gives the space of equivalence classes of deformations.

In [28] and [10], we proved the following theorem.

Theorem 4.10 *Let (P_1, P_2) be a semisimple bihamiltonian structure, then*

$$BH^2_{d\geq 2}(\hat{\mathcal{F}}, d_1, d_2) \cong \begin{cases} \bigoplus_{i=1}^{n} C^\infty(\mathbb{R}), & d = 3; \\ 0, & d = 2, 4, 5, \dots. \end{cases}$$

The classes in $BH^2_3(\hat{\mathcal{F}})$ are actually parameterized by n functions of canonical coordinates $\{c_1(\lambda_1), \dots, c_n(\lambda_n)\}$. We will discuss their definition and properties in the next section.

In [30], we prove the following theorem.

Theorem 4.11 *Let $\big(P_1 = \int (\theta\theta')\,\mathrm{d}x,\ P_2 = \int (u\theta\theta')\,\mathrm{d}x\big)$ be the leading term of the bihamiltonian structure of KdV equation, then*

$$BH^3_{d\geq 4}(\hat{\mathcal{F}}, d_1, d_2) \cong 0.$$

The proofs of these two theorems are very long, so they cannot be given here. Combining the above theorems and lemma, we obtain the following corollaries.

Corollary 4.12 *Let (P_1, P_2) be a semisimple bihamiltonian structure. For any deformation $(\tilde{P}_1, \tilde{P}_2)$ of (P_1, P_2), one can define n functions*

$$c_1(\lambda_1), \dots, c_n(\lambda_n),$$

which are called the central invariants of $(\tilde{P}_1, \tilde{P}_2)$, such that

(a) Two deformations are equivalent if and only if their central invariants coincide.

(b) Write the deformation $(\tilde{P}_1, \tilde{P}_2)$ as the sum of homogeneous components

$$\tilde{P}_a = P_a + \sum_{k \geq 1} \varepsilon^k P_a^{[k]}, \quad a = 1, 2,$$

where $P_a^{[k]} \in \hat{\mathcal{F}}^2_{k+1}$, then there exists a gauge transformation e^{ad_Z} such that $(e^{\mathrm{ad}_Z}(\tilde{P}_1), e^{\mathrm{ad}_Z}(\tilde{P}_2))$ doesn't contain odd powers of ε.

(c) If (P_1, P_2) is the leading term of the bihamiltonian structure of the KdV hierarchy, then for any smooth function $c(u)$ there exists a deformation whose central invariant is given by $c(u)$.

Part (a) is called the uniqueness theorem of the deformation problem. Part (b) is important for Gromov-Witten theory, because it ensure that the corresponding integrable hierarchy can be always written as a formal power series of $\hbar$. Part (c) is called the existence theorem of the deformation problem. We conjecture that it is true for arbitrary semisimple bihamiltonian structure.

Recently [2] (c.f. [1]), Carlet, Posthuma, and Shadrin proved the following theorem, which showed that our conjecture is true.

Theorem 4.13 ([2]) *Let (P_1, P_2) be a semisimple bihamiltonian structure of hydrodynamic type, then $BH^p_d(\hat{\mathcal{F}}, d_1, d_2)$ vanishes for most (p, d). In particular, $BH^3_{d\geq 5}(\hat{\mathcal{F}}, d_1, d_2) \cong 0$, which implies that the existence of a full dispersive deformation of (P_1, P_2) starting from any its infinitesimal deformation.*

The proof of this theorem is sophisticated, so we cannot give it here. Please refer to [2] for details.

4.3 Bihamiltonian vector fields

Let (P_1, P_2) be a bihamiltonian structure, $X \in \hat{\mathcal{F}}^1$ is called a bihamiltonian vector field, if there exists $I, J \in \hat{\mathcal{F}}^0$ such that $X = d_1(I) = d_2(J)$. Suppose (P_1, P_2) is semisimple, and $(\tilde{P}_1, \tilde{P}_2)$ is a deformation of (P_1, P_2). In this subsection, we will consider their bihamiltonian vector fields.

Lemma 4.14 *The space of bihamiltonian vector fields of (P_1, P_2) is given by $BH^1_{\geq 1}(\hat{\mathcal{F}}, d_1, d_2)$.*

Proof: Let X be a bihamiltonian vector field of (P_1, P_2). Note that $\deg(P_a) = 1$ $(a = 1, 2)$, so $\nu(X) \geq 1$.

The bihamiltonian cohomology $BH^1_{\geq 1}(\hat{\mathcal{F}}, d_1, d_2)$ is defined as

$$BH^1_{\geq 1}(\hat{\mathcal{F}}, d_1, d_2) = \{X \in \hat{\mathcal{F}}^1_{\geq 1} \mid d_1(X) = 0,\ d_2(X) = 0\},$$

so every bihamiltonian vector field belongs to $BH^1_{\geq 1}(\hat{\mathcal{F}}, d_1, d_2)$.

On the other hand, if $X \in BH^1_{\geq 1}(\hat{\mathcal{F}}, d_1, d_2)$, then there must exist $I, J \in \hat{\mathcal{F}}^0$ such that $X = d_1(I) = d_2(J)$, because $H^1_{\geq 1}(\hat{\mathcal{F}}, d_a) \cong 0$ $(a = 1, 2)$. □

Lemma 4.15 *We have* $BH^1_{\geq 2}(\hat{\mathcal{F}}, d_1, d_2) \cong 0$.

Proof: Suppose $X = d_1(I) = d_2(J) \in \hat{\mathcal{F}}^1_d$ $(d \geq 2)$, where

$$I = \int p\mathrm{d}x, \quad J = \int q\mathrm{d}x,$$

and $p, q \in \mathcal{A}^{(N)}$, $1 \leq N \leq d$. We are to show that one can always choose another pair of density $p', q' \in \mathcal{A}^{(N-1)}$ such that $I = \int p'\mathrm{d}x$, $J = \int q'\mathrm{d}x$. Then the theorem can be proved by induction on N.

Let $Z = d_1(I) - d_2(J) = \int (Z^\alpha \theta_\alpha)\,\mathrm{d}x$. It is easy to see that $Z^\alpha \in \mathcal{A}^{(2N+1)}$. We introduce a notation $a_{(i,s)} = \frac{\partial a}{\partial \lambda^{i,s}}$ for $a \in \mathcal{A}$. Then one can obtain that

$$Z^i_{(j,2N+1)} = (-1)^{N+1} f^i \left(p_{(i,N)(j,N)} - \lambda^i q_{(i,N)(j,N)}\right) = 0,$$

so $(\lambda^i - \lambda^j) q_{(i,N)(j,N)} = 0$. Since $\lambda^i \neq \lambda^j$ $(i \neq j)$, we have $q_{(i,N)(j,N)} = 0$ $(i \neq j)$. Denote by $r^i = q_{(i,N)(i,N)}$, then $q_{(i,N)(j,N)} = \delta^{ij} r^i$, $p_{(i,N)(j,N)} = \delta^{ij}\lambda^i r^i$.

Next, compute $Z^i_{(j,2N)}$:

$$0 = Z^i_{(j,2N)} = (-1)^{N+1} \left(\left(N + \frac{1}{2}\right) f^i r^i \lambda^{i,1} \delta^{ij} + (\lambda^i A^{ij} - B^{ij}) + \right.$$
$$\left. f^i \left(p_{(i,N)(j,N-1)} - p_{(j,N)(i,N-1)}\right) - g^i \left(q_{(i,N)(j,N-1)} - q_{(j,N)(i,N-1)}\right)\right).$$

Take $i = j$, we obtain $r^i = 0$, so p and q are linear in $\lambda^{\alpha,N}$. For $i \neq j$, we obtain

$$p_{(i,N)(j,N-1)} = p_{(j,N)(i,N-1)}, \quad q_{(i,N)(j,N-1)} = q_{(j,N)(i,N-1)},$$

which imply that one can choose $\tilde{p}, \tilde{q} \in \mathcal{A}^{(N-1)}$ such that $p' = p - \partial(\tilde{p})$ and $q' = q - \partial(\tilde{q})$ belong to $\mathcal{A}^{(N-1)}$. The lemma is proved. □

The above proof can be regarded as a demo version of the proofs for Theorem 4.10 and 4.11. In the latter cases, we also use an induction on N for $\mathcal{A}^{(N)}$. This computation method can be translated to the language of spectral sequence. In Carlet, Posthuma, and Shadrin's new preprints [1, 2], they introduce more spectral sequences, which help them to compute almost all the bihamiltonian cohomologies $BH^p_d(\hat{\mathcal{F}}, d_1, d_2)$.

The above lemma shows that bihamiltonian vector fields of (P_1, P_2) must have degree one.

Corollary 4.16 *Let*

$$X = \int (X^\alpha \theta_\alpha)\,\mathrm{d}x$$

be a bihamiltonian vector fields of (P_1, P_2)*, then* X *must be diagonal hydrodynamic, i.e.* $X^i = V^i(\lambda)\lambda^{i,1}$.

Proof: Suppose $X = d_1(I) = d_2(J)$, where $I = \int p\mathrm{d}x$, $J = \int q\mathrm{d}x$, $p, q \in \mathcal{A}_0$, then X^i takes the following form:

$$X^i = \sum_{j=1}^{n} V^i_j(\lambda)\lambda^{j,1},$$

where the coefficients read

$$V^i_j = -f^i \mathcal{D}_{ij}(p) = -g^i \mathcal{D}_{ij}(q),$$

and $\mathcal{D}_{ij}$ is the following linear differential operator

$$\mathcal{D}_{ij} = \frac{\partial^2}{\partial\lambda^i\partial\lambda^j} + \frac{1}{2}\frac{\partial \log f^i}{\partial\lambda^j}\frac{\partial}{\partial\lambda^i} + \frac{1}{2}\frac{\partial \log f^j}{\partial\lambda^i}\frac{\partial}{\partial\lambda^j}.$$

Note that $\mathcal{D}_{ij}$ is symmetric, so we have $(\lambda^i - \lambda^j)\mathcal{D}_{ij}(q) = 0$, which implies $V^i_j = 0$ if $i \neq j$. The corollary is proved. □

Corollary 4.17 *If X_1, X_2 are bihamiltonian vector fields of (P_1, P_2), then $[X_1, X_2] = 0$.*

Proof: Let $Y = [X_1, X_2]$, then $d_1(Y) = 0$, $d_2(Y) = 0$, so $Y = 0$, since $Y \in BH^1_2(\hat{\mathcal{F}}, d_1, d_2) \cong 0$. □

Now let us consider the bihamiltonian vector fields of $(\tilde{P}_1, \tilde{P}_2)$. Let $X \in \hat{\mathcal{F}}^1$ be such a vector field, then $\nu(X) \geq 1$. We expand it with respect the standard gradation

$$X = X_1 + X_2 + \ldots, \quad X_d \in \hat{\mathcal{F}}^1_d,$$

then it is easy to see that X_1 must be a bihamiltonian vector field of (P_1, P_2). We call X_1 the leading term of X.

Theorem 4.18 *(a) If X_1, X_2 are bihamiltonian vector fields of $(\tilde{P}_1, \tilde{P}_2)$, then $[X_1, X_2] = 0$. If they have the same leading term, then $X_1 = X_2$.*

(b) For any bihamiltonian vector field X_1 of (P_1, P_2), there exists a bihamiltonian vector field X of $(\tilde{P}_1, \tilde{P}_2)$ such that X's leading term is just X_1.

Proof: For Part (a), we only need to show that if the leading term of a bihamiltonian vector field X of $(\tilde{P}_1, \tilde{P}_2)$ vanishes, then $X = 0$. Expand X as

$$X = X_1 + X_2 + X_3 + \cdots, \quad X_1 = 0, \ X_d \in \hat{\mathcal{F}}^1_d.$$

We also expand $(\tilde{P}_1, \tilde{P}_2)$ in the same way:

$$\tilde{P}_1 = P_1 + \sum_{k\geq 1} P_1^{[k]}, \quad \tilde{P}_2 = P_2 + \sum_{k\geq 1} P_2^{[k]}.$$

Then the condition $[\tilde{P}_a, X] = 0$ $(a = 1, 2)$ implies that

$$d_a(X_d) + \sum_{k=1}^{d-2} [P_a^{[k]}, X_{d-k}] = 0, \quad a = 1, 2.$$

When $d = 2$, we obtain $d_1(X_2) = 0$, $d_2(X_2) = 0$, so we have $X_2 = 0$. Then, by induction on d, one can show that $X_d = 0$ for $d = 2, 3, \ldots$, so $X = 0$.

To prove Part (b), we also expand X, P_1, and P_2 as above. We need to show that if X_1 satisfies $d_1(X_1) = 0$, $d_2(X_1) = 0$, then there exist $X_2, X_3, \ldots$ such that

$$d_a(X_d) + \sum_{k=1}^{d-1} [P_a^{[k]}, X_{d-k}] = 0, \quad a = 1, 2. \tag{4.1}$$

Without loss of generality, we can assume that $P_a^{[1]} = 0$ (see Part (b) of Corollary 4.12), then we can take $X_2 = 0$ directly.

The existence of X_d ($d \geq 3$) can be proved by induction on d. Suppose we have obtained $X_2, \ldots, X_{d-1}$, and we are to find X_d. Denote by

$$W_a = -\sum_{k=1}^{d-1} [P_a^{[k]}, X_{d-k}], \quad a = 1, 2,$$

then X_d satisfy $d_1(X_d) = W_1$ and $d_2(X_d) = W_2$.

We assert that $d_1(W_1) = 0$. By using the Jacobi identity, we have

$$d_1(W_1) = \sum_{k=1}^{d-1} \left([d_1(P_1^{[k]}), X_{d-k}] + [P_1^{[k]}, d_1(X_{d-k})] \right).$$

Note that $\tilde{P}_1$ is a Hamiltonian structure, so we have

$$d_1(P_1^{[k]}) + \frac{1}{2} \sum_{j=1}^{k-1} [P_1^{[j]}, P_1^{[k-j]}] = 0.$$

From the above identity and (4.1) with X_d replaced by X_{d-k}, one can show that $d_1(W_1) = 0$. Similarly, we have $d_2(W_2) = 0$.

Since $H_{d+1}^2(\hat{\mathcal{F}}, d_1) \cong 0$, there exists $Y \in \hat{\mathcal{F}}_d^1$ such that $W_1 = d_1(Y)$, then the general solution to $d_1(X_d) = W_1$ can be written as $X_d = Y + d_1(Z)$ for arbitrary $Z \in \hat{\mathcal{F}}_{d-1}^0$. Then the equation $d_2(X_d) = W_2$ becomes $d_1 d_2(Z) = Q$, where $Q = d_2(Y) - W_2$.

It is easy to see that $d_2(Q) = 0$. One can also show that $d_1(Q) = 0$ by using the condition $[\tilde{P}_1, \tilde{P}_2] = 0$, so $Q \in \hat{\mathcal{F}}_{d+1}^2 \cap \mathrm{Ker}(d_1) \cap \mathrm{Ker}(d_2)$. Note that $d+1 \geq 4$, so $BH_d^2(\hat{\mathcal{F}}, d_1, d_2) \cong 0$, so there must exist $Z \in \hat{\mathcal{F}}_{d-1}^0$ such that $Q = d_1 d_2(Z)$. The existence of X_d is proved. □

5 Central Invariants

5.1 Definition and properties

In this subsection, we explain how to compute the central invariants of a deformed semisimple bihamiltonian structure.

Let (P_1, P_2) be a semisimple bihamiltonian structure, $(\tilde{P}_1, \tilde{P}_2)$ be a deformation of (P_1, P_2), and $\mathcal{P}_a, \tilde{\mathcal{P}}_a$ ($a = 1, 2$) be the corresponding matrix differential operators in canonical coordinates. Expand $\tilde{\mathcal{P}}_a$ ($a = 1, 2$) with respect to the standard gradation

$$\tilde{\mathcal{P}}_a^{\alpha\beta} = \mathcal{P}_a^{\alpha\beta} + \sum_{s \geq 1} \left(\sum_{t=0}^{s+1} P_{s,t,a}^{\alpha\beta} \partial^t \right),$$

where $a = 1, 2$, $P_{s,t,a}^{\alpha\beta} \in \mathcal{A}_{s+1-t}$. It is easy to see that $P_{s,s+1,a}^{\alpha\beta}$ is a tensor on M. The *central invariants* of $(\tilde{P}_1, \tilde{P}_2)$ are defined as

$$c_i(\lambda) = \frac{1}{3\,(f^i)^2} \left(P_{2,3,2}^{ii} - \lambda^i P_{2,3,1}^{ii} + \sum_{k \neq i} \frac{\left(P_{1,2,2}^{ki} - \lambda^i P_{1,2,1}^{ki}\right)^2}{f^k(\lambda^k - \lambda^i)} \right), \tag{5.1}$$

where $i = 1, \dots, n$, λ_i's are the canonical coordinates, and f^i's are the diagonal entries of the first metric (see Definition 4.5 and Theorem 4.6). Note that the semisimplicity of (P_1, P_2) plays a crucial role in the definition of central invariants: (i) λ_i's are not constants, so we can use them as coordinates; (ii) they are distinct, so the denominator in the above formula never vanish.

Theorem 5.1 *(a) The central invariants are invariant under gauge transformations.*

(b) The i-th central invariant $c_i(\lambda)$ only depends on λ^i.

(c) The cohomology class corresponding to the infinitesimal deformation of $(\tilde{P}_1, \tilde{P}_2)$ has a representative

$$Q = d_2 d_1 \left(\int \left(\sum_{i=1}^{n} c_i(\lambda^i) \lambda^{i,1} \log \lambda^{i,1} \right) \mathrm{d}x \right) \in \hat{\mathcal{F}}^2_3.$$

The proof of this theorem is simple but tedious [10], so we omit it.

In Part (c), we give Q in the form $d_2 d_1(J)$. This expression looks confusing, since elements of the form $d_2 d_1(J) = -d_1 d_2(J)$ should be exact in the cohomology group $BH^2_3(\hat{\mathcal{F}}, d_1, d_2)$. But Q is indeed not trivial, because the density of the local functional J given above is *not* a differential polynomial, so $J \notin \hat{\mathcal{F}}^0$. This expression shows that if we enlarge the group of gauge transformation, then there is no nontrivial infinitesimal deformations. This result is called the quasi-triviality theorem [10].

Theorem 5.2 *Denote by $\mu = \prod_{i=1}^n \lambda^{i,1}$, $\tilde{\mathcal{A}} = \hat{\mathcal{A}}[\mu^{-1}]$, $\tilde{\mathcal{F}} = \tilde{\mathcal{A}}/\partial\tilde{\mathcal{A}}$.*

(a) For any deformation $(\tilde{P}_1, \tilde{P}_2)$ of a semisimple bihamiltonian structure (P_1, P_2), there exists $Z \in \tilde{\mathcal{F}}^1_{>0}$, such that $(e^{\mathrm{ad}_Z}(\tilde{P}_1), e^{\mathrm{ad}_Z}(\tilde{P}_2)) = (P_1, P_2)$.

(b) Let $X \in \hat{\mathcal{F}}^1$ be a bihamiltonian vector field of $(\tilde{P}_1, \tilde{P}_2)$ with leading term $X_1 \in \hat{\mathcal{F}}^1_1$, then $e^{\mathrm{ad}_Z}(X) = X_1$.

This theorem implies that Dubrovin-Zhang's QT Axiom is a corollary of the BH Axiom, so the QT Axiom can be removed from their construction.

5.2 Example: Frobenius manifolds

Let (P_1, P_2) be the bihamiltonian structure associated to a semisimple Frobenius manifold (see Example 4.7). In [13], Dubrovin and Zhang constructed a genus one deformation of (P_1, P_2) satisfying their VS Axiom [14]. Note that a genus one deformation is exactly an infinitesimal deformation of degree 3. So it is natural to ask: what are its central invariants?

By checking the expressions given in [13], the tensors used in (5.1) read

$$f^i = \frac{1}{\psi_{i1}^2}, \quad P^{ki}_{1,2,1} = 0, \quad P^{ki}_{1,2,2} = 0,$$

$$P^{ii}_{2,3,1} = \frac{1}{12\psi_{i1}^4} \sum_{j \neq i} \gamma_{ij} \left(\frac{\psi_{i1}}{\psi_{j1}} + \frac{\psi_{j1}}{\psi_{i1}} \right),$$

$$P^{ii}_{2,3,2} = \frac{1}{72\psi_{i1}^4} + \frac{\lambda^i}{12\psi_{i1}^4} \sum_{j \neq i} \gamma_{ij} \left(\frac{\psi_{i1}}{\psi_{j1}} + \frac{\psi_{j1}}{\psi_{i1}} \right),$$

then we immediately obtain the central invariants

$$c_1 = \cdots = c_n = \frac{1}{24}.$$

In [36], Zhang showed that if a deformation $(\tilde{P}_1, \tilde{P}_2)$ admits a tau function, then its central invariants must be constant. In this case, the genus one free energy has the form

$$F_1 = \sum_{i=1}^{n} c_i \log(\lambda^{i,1}) + G(\lambda).$$

When $c_i = 1/24$ $(i = 1, \ldots, n)$, we obtain the well-known formula for genus one free energy of a semisimple cohomological field theory

$$F_1 = \frac{1}{24} \log\left(\prod_{i=1}^{n} \lambda^{i,1}\right) + G(\lambda).$$

We conjecture that the converse propositions of the above results are also true.

Conjecture 5.3 *Let $(\tilde{P}_1, \tilde{P}_2)$ be a deformation of (P_1, P_2) with central invariants $c_1, \ldots, c_n$.*

(a) If c_i $(i = 1, \ldots, n)$ are all constant, then the corresponding integrable hierarchy admits a tau structure.

(b) if $c_i = 1/24$ $(i = 1, \ldots, n)$, then the corresponding integrable hierarchy has linearizable Virasoro symmetries.

If these conjectures hold true, then Dubrovin-Zhang's TS Axiom and VS Axiom can be replaced by the above conditions on central invariants.

5.3 Example: Drinfeld-Sokolov hierarchy

Let $\mathfrak{g}$ be a simple Lie algebra of dimension m and rank n, and $\mathrm{u}^1, \ldots, \mathrm{u}^m$ be a set of basis. Suppose

$$[\mathrm{u}^\alpha, \mathrm{u}^\beta] = C^{\alpha\beta}_\gamma \mathrm{u}^\gamma.$$

Let $M = \mathfrak{g}^*$, and $\mathrm{v}_1, \ldots, \mathrm{v}_m$ be dual basis of $\mathrm{u}^1, \ldots, \mathrm{u}^m$, then any element $q \in M$ can be written as

$$q = u^\alpha \mathrm{v}_\alpha, \quad u^\alpha \in \mathbb{R}.$$

The bracket

$$\{u^\alpha, u^\beta\} = C^{\alpha\beta}_\gamma u^\gamma$$

defines a Poisson structure on M, which is called the Lie-Poisson structure.

The Lie-Poisson structure defines a Hamiltonian structure $P_0 \in \hat{\mathcal{F}}^2_0$:

$$P_0 = \int \left(C^{\alpha\beta}_\gamma u^\gamma \theta_\alpha \theta_\beta\right) \mathrm{d}x.$$

Its action on $F, G \in \hat{\mathcal{F}}^0$ is given by

$$\{F, G\}_{P_0} = \int \left(C^{\alpha\beta}_\gamma u^\gamma \frac{\delta F}{\delta u^\alpha} \frac{\delta G}{\delta u^\beta}\right) \mathrm{d}x. \tag{5.2}$$

Note that $C^{\alpha\beta}_{\gamma}u^{\gamma} = \langle q, [\mathrm{u}^{\alpha}, \mathrm{u}^{\beta}]\rangle$, where $\langle\,,\rangle$ is the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. If we introduce a notation

$$\mathrm{grad}(F) = \frac{\delta F}{\delta u^{\alpha}}\mathrm{u}^{\alpha} \in \mathcal{A}\otimes\mathfrak{g},$$

then the Poisson bracket (5.2) becomes

$$\{F, G\}_{P_0} = \int \langle q, [\mathrm{grad}(F), \mathrm{grad}(G)]\rangle \mathrm{d}x.$$

Let $\langle\,,\rangle_{\mathfrak{g}}$ be a invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$, we can identify $\mathfrak{g}^*$ and g such that

$$\langle q, \cdot\rangle = \langle q, \cdot\rangle_{\mathfrak{g}}.$$

We always assume this identification, then the Poisson bracket (5.2) can be also written as

$$\{F, G\}_{P_0} = \int \langle \mathrm{grad}(F), [\mathrm{grad}(G), q]\rangle_{\mathfrak{g}} \mathrm{d}x.$$

Define $\eta^{\alpha\beta} = \langle \mathrm{u}^{\alpha}, \mathrm{u}^{\beta}\rangle_{\mathfrak{g}}$. The Hamiltonian structure P_0 admit a degree one deformation

$$P = \int \left(C^{\alpha\beta}_{\gamma}u^{\gamma}\theta_{\alpha}\theta_{\beta} - \eta^{\alpha\beta}\theta_{\alpha}\theta^1_{\beta}\right)\mathrm{d}x.$$

The action of P on $F, G \in \hat{\mathcal{F}}^0$ reads

$$\{F, G\}_{P} = \int \langle \mathrm{grad}(F), [\mathrm{grad}(G), \partial + q]\rangle_{\mathfrak{g}} \mathrm{d}x.$$

Here we assume that $[\partial, a] = -[a, \partial] = \partial(a)$ for $a \in \mathcal{A}\otimes\mathfrak{g}$.

Let $X_0 = \int (u^{\alpha}_0\theta_{\alpha})\,\mathrm{d}x$, where $u^1_0, \dots, u^m_0 \in \mathbb{R}$ are some fixed constants. Then it is easy to see that $[X_0, [X_0, P]] = 0$, so $([X_0, P], P)$ forms an exact bihamiltonian structure. We rename $P_1 = [X_0, P]$, $P_2 = P$. The bihamiltonian structure (P_1, P_2) is called the Zakharov-Shabat bihamiltonian structure.

The second component of the Zakharov-Shabat bihamiltonian structure can be regarded as a reduction of the deformed Hamiltonian structure mentioned in Example 3.8. The Drinfeld-Sokolov bihamiltonian structure is a further reduction of the Zakharov-Shabat one. A detailed description of the Drinfeld-Sokolov bihamiltonian structure would make the present lecture notes too long, so we only give the final result.

Theorem 5.4 ([11]) *The Drinfeld-Sokolov bihamiltonian structure* (Q_1, Q_2) *is an exact bihamiltonian structure on a submanifold* $V \subset M$ *with* $\dim V = n$.

(a) The leading term of (Q_1, Q_2) *coincides with the bihamiltonian structure associated to the Frobenius structure on the orbit space of the Weyl group of* $\mathfrak{g}$. *In particular, it is semisimple.*

(b) The central invariants of (Q_1, Q_2) *are given by (up to a rearrangement)*

$$c_i = \frac{\langle \alpha_i^{\vee}, \alpha_i^{\vee}\rangle_{\mathfrak{g}}}{48}, \quad i = 1, \dots, n,$$

where $\{\alpha_1^{\vee}, \dots, \alpha_n^{\vee}\}$ *is a collection of simple coroots of* $\mathfrak{g}$.

(c) If we choose $\langle\,,\rangle_{\mathfrak{g}}$ *to be the normalized one*

$$\langle\,,\rangle_{\mathfrak{g}} = \frac{1}{2h^{\vee}}\langle\,,\rangle_{\mathrm{K}},$$

where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$, and $\langle\,,\rangle_{\mathrm{K}}$ is the Killing form, then the central invariants for $\mathfrak{g}$ of type X_n is given by the following table.

$$
\begin{array}{lcccc}
X_n & c_1 & \cdots & c_{n-1} & c_n \\
A_n & \frac{1}{24} & \cdots & \frac{1}{24} & \frac{1}{24} \\
B_n & \frac{1}{24} & \cdots & \frac{1}{24} & \frac{1}{12} \\
C_n & \frac{1}{12} & \cdots & \frac{1}{12} & \frac{1}{24} \\
D_n & \frac{1}{24} & \cdots & \frac{1}{24} & \frac{1}{24} \\
E_n,\ n=6,7,8 & \frac{1}{24} & \cdots & \frac{1}{24} & \frac{1}{24} \\
F_n,\ n=4 & \frac{1}{24} & \frac{1}{24} & \frac{1}{12} & \frac{1}{12} \\
G_n,\ n=2 & \frac{1}{8} & & & \frac{1}{24}
\end{array}
\tag{5.3}
$$

When $\mathfrak{g}$ is of ADE type, the central invariants are all equal to $1/24$, so the Drinfeld-Sokolov bihamiltonian structure is equivalent to Dubrovin-Zhang's deformation [13, 15], and the total descendant potential coincides with the one given by Givental's formula. Recently, Fan, Jarvis and Ruan rigorously define the Landau-Ginzburg A-model for a quasi-homogeneous singularity, which is called the FJRW theory. They also proved that the total descendant potential of FJRW theory for an ADE singularity is given by Givental's formula, so it is a tau function of the corresponding Drinfeld-Sokolov hierarchy. This result is called the ADE Witten conjecture. Please see [18, 19, 26, 21, 33, 34, 27] for more details.

When $\mathfrak{g}$ is of $BCFG$ type, the central invariants are constant, but not all equal to $1/24$. Define $R = 24\sum_{i=1}^{n} c_i$, then we have

$\mathfrak{g}$	B_n	C_n	F_4	G_2
R	$n+1$	$2n-1$	6	4

It is well-known that a simple Lie algebra of B_n type can be embedded into a simple Lie algebra of D_{n+1} type as the fixed locus of an order two automorphism. Similarly, C_n can be embedded into A_{2n-1}, F_4 can be embedded into E_6, and G_2 can be embedded into D_4. So the number R gives exactly the rank of the *ambient* Lie algebra. This observation suggests us how to prove the generalized Witten conjecture of $BCFG$ type [27].

Remark 5.5 *The above two examples both have constant central invariants. There also exist bihamiltonian structures possessing non-constant central invariats. For example, the bihamiltonian structure of the Camassa-Holm hierarchy (see Example 3.2) has $c(\lambda) = \frac{\lambda}{3}$. Its two-component generalization (see [28], [3]) has*

$$c_1(\lambda_1) = \frac{\lambda_1^2}{24}, \quad c_2(\lambda_2) = \frac{\lambda_2^2}{24}.$$

We also considered its multi-components generalization in [4], and more complicated central invariants arose there.

The Camassa-Holm equation and its generalization are very popular recently in the area like PDE analysis or hydrodynamic, because they often have interesting weak solutions and wave-breaking phenomena. They are also the main source of our work [29], whose results play an important role in the present paper. However, there seems no direct connection between such integrable hierarchies and Gromov-Witten theories.

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