数学分析（3）：第 4 次习题课

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1.（Raabe）设 $\alpha > 0$，计算

$$I(\alpha) = \int_{\alpha}^{\alpha+1} \log \Gamma(s) \mathrm{d}s.$$

2. 设 n 是正整数，求证：

   i)（Euler）

$$\prod_{k=1}^{n} \Gamma\left(\frac{k}{n}\right) = \frac{(2\,\pi)^{\frac{n-1}{2}}}{\sqrt{n}}.$$

   ii)（Gauss）

$$\prod_{k=0}^{n-1} \Gamma\left(\alpha + \frac{k}{n}\right) = \frac{(2\,\pi)^{\frac{n-1}{2}}}{n^{n\,\alpha+\frac{1}{2}}} \Gamma(n\,\alpha).$$

3. 设 $\alpha > 0$，求证：

   i)

$$\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} = -\gamma + \sum_{k=0}^{\infty} \left(\frac{1}{k+1} - \frac{1}{k+\alpha}\right).$$

   其中 γ 是 Euler 常数：

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right).$$

   ii)

$$\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} = -\gamma + \int_{0}^{1} \frac{1-t^{\alpha-1}}{1-t} \mathrm{d}t.$$

   iii)

$$\Gamma(\alpha+1) = e^{-\gamma\,\alpha} \prod_{n=1}^{\infty} \left(1 + \frac{\alpha}{n}\right)^{-1} e^{\frac{\alpha}{n}}.$$

   iv)

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - \alpha^2} = \frac{1 - \pi\,\alpha\,\cot \pi\,\alpha}{2\,\alpha^2},$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + \alpha^2} = \frac{\pi\,\alpha\,\coth \pi\,\alpha - 1}{2\,\alpha^2}.$$

v)

$$\zeta(2)=\frac{\pi^2}{6},\quad \zeta(4)=\frac{\pi^4}{90},\quad \zeta(6)=\frac{\pi^6}{945},\quad \ldots$$

4. 证明或计算:

i)

$$\begin{aligned}
&\Gamma'(1)=-\gamma,\quad \Gamma'\left(\frac{1}{2}\right)=-\sqrt{\pi}(\gamma+2\log 2),\\
&\Gamma''(1)=\gamma^2+\frac{\pi^2}{6},\quad \Gamma'\left(\frac{1}{2}\right)=\sqrt{\pi}\left(\frac{\pi^2}{2}+(\gamma+2\log 2)^2\right),\\
&\Gamma'''(1)=-\gamma^3-\gamma\frac{\pi^2}{2}-2\,\zeta(3),\quad \Gamma'''\left(\frac{1}{2}\right)=?
\end{aligned}$$

ii)

$$\begin{aligned}
&\int_0^\infty \frac{\sin x}{x}\log x\,\mathrm{d}x=-\gamma\frac{\pi}{2},\\
&\int_0^\infty \frac{\sin x}{x}\log^2 x\,\mathrm{d}x=\gamma^2\frac{\pi}{2}+\frac{\pi^3}{24},\\
&\int_0^\infty \frac{\sin x}{x}\log^3 x\,\mathrm{d}x=?
\end{aligned}$$